Trigonometry Workbook Solutions

Inverse trig functions

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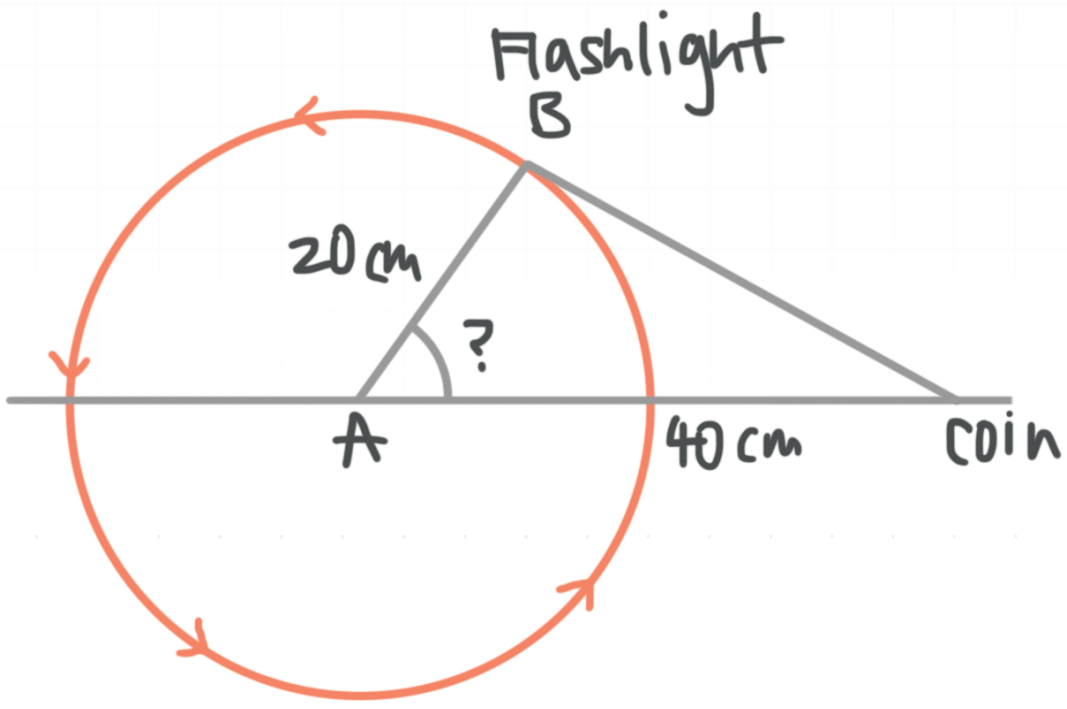
MATH

INVERSE TRIG RELATIONS

■ 1. A stick AB of length 20 cm rotates counterclockwise about the endpoint A with a constant speed, and makes one revolution every 20 seconds. A laser flashlight is fixed on the endpoint B of the stick, in such a way that the light shines orthogonally to the stick. A coin is placed 40 cm directly to the right of A. Find the set of all possible values of θ between the stick and the line that connects A to the coin, such that the flashlight points exactly to the coin.



Solution:

Consider the position of the stick when the flashlight points exactly to the coin.

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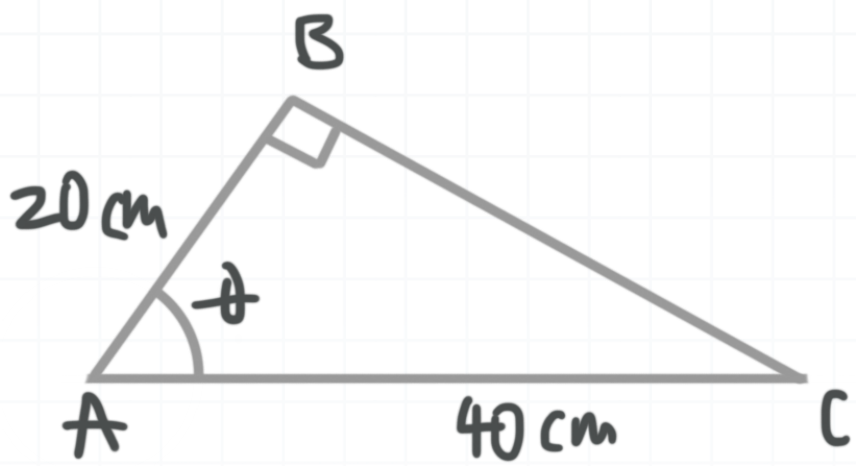
Let the point C be the position of the coin. Since a light shines orthogonally to the stick, the angle B is right, and so the triangle ABC is a right triangle.



In this triangle, the angle $A = \theta$ is the angle we're interested in, the leg AB is equal to 20 cm, and hypotenuse AC is equal to 40 cm. So by the definition of the cosine of an acute angle in a right triangle,

$$\cos\theta = \frac{AB}{AC} = \frac{20}{40} = \frac{1}{2}$$

So we need to find the set of all possible values θ where

$$\cos\theta = \frac{1}{2}$$

The flashlight points to the coin only for the angles θ in quadrant I. So $\theta = \pi/3$. Extend the solution to other periods, and then we can say

$$\theta = \frac{\pi}{3} + 2\pi n, \text{ where } n \text{ is any integer}$$

■ 2. A beacon is anchored to the seabed such that it can move freely only in the vertical direction. Because of waves, the beacon's altitude with

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respect to the ground is changed periodically and can be modeled by $h(t) = 16 + 0.8\cos(0.4t)$, where t is time in seconds. Find the moments in time when the distance from the beacon to the seabed is the largest.

Solution:

Since the cosine function has range of $[-1,1]$, the beacon reaches its largest altitude of 16.8 at the moments in time when $\cos(0.4t)$ reaches its maximum value of 1.

So we need to find the set of all possible values of angles t such that

$$\cos(0.4t) = 1$$

The equation $\cos\phi = 1$ is true when $\phi = 0$ in $[0,2\pi)$. Since the cosine function has period 2π, we can get other solutions by adding multiples of 2π. Let $\phi = 0.4t$. We need to find $\phi \in [0,2\pi)$ such that

$$\cos\phi = 1$$

In the unit circle, the cosine of ϕ is equal to the x-coordinate of P at which the terminal side of ϕ intersects the unit circle. Sketch the unit circle and the line $x = 1$. The point where P intersects the circle has an x-coordinate of 1, so the corresponding angle is the solution of the equation.

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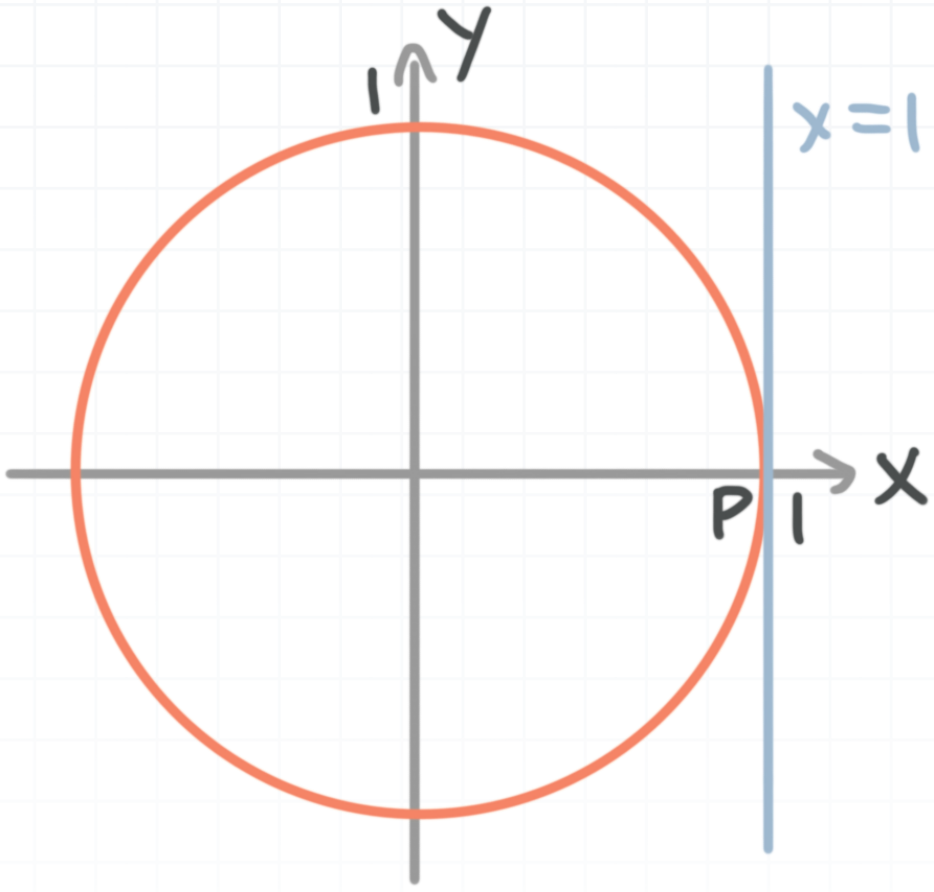

The angle we're interested in is $\phi_0 = 0$. Extend the solution to other periods by adding multiples of 2π. So the solutions are

$\phi_0 = 0 + 2\pi n$, where n is any integer

Substitute $\phi_0 = 0.4x$ and solve the equation for x.

$$0.4x = 2\pi n$$

$$x = \frac{2\pi}{0.4}n$$

$$x = 2\pi \cdot \frac{5}{2}n$$

$x = 5\pi n$, where n is any integer

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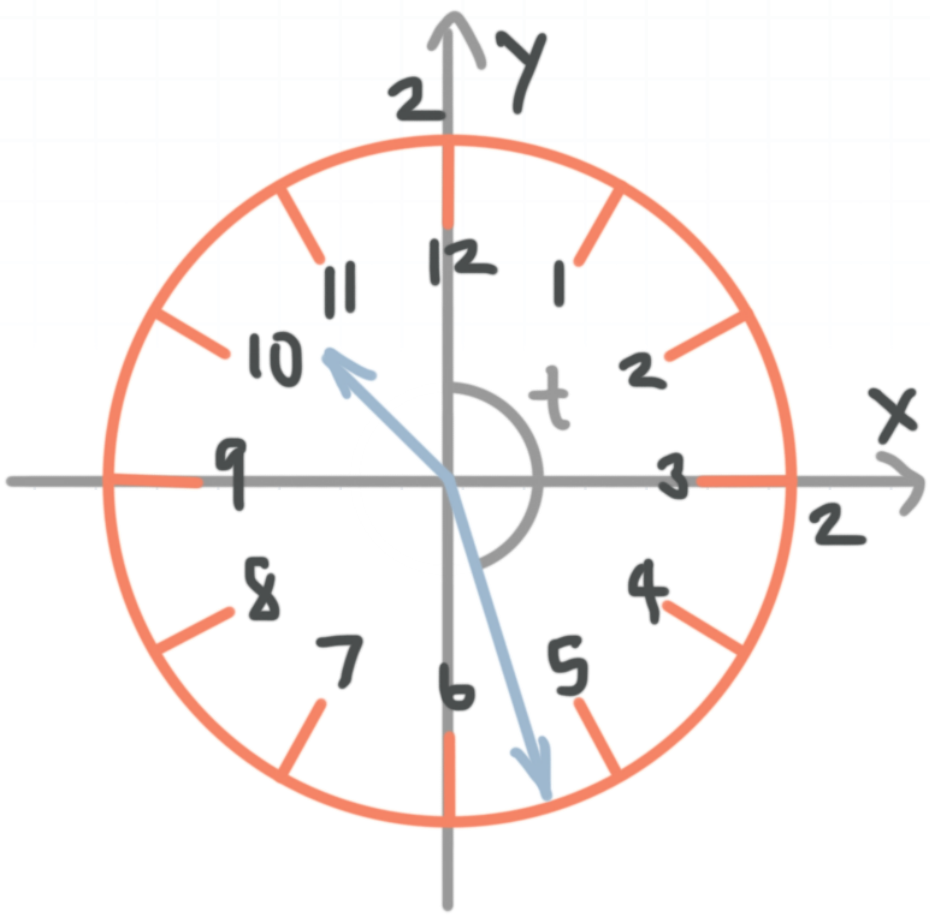
■ 3. The minute hand of a 12-hour clock makes one revolution every hour, and has a length of 2 cm. Find the equations of $x(t)$ and $y(t)$ that model the minute-hand motion in the form

$$x(t) = a\sin(bt)$$

$$y(t) = a\cos(bt)$$

where t is the angle between the minute hand and 12 h direction, and $x(t)$, $y(t)$ are the coordinates of the minute hand endpoint with respect to the center of the clock dial. Then find the set of all possible values of time when $x(t) = 1$.



Solution:

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Since the minute hand has a length of 2 cm, the values of $x(t)$ and $y(t)$ have range $[-2,2]$. So the amplitude is 2, and also the parameter $a = 2$.

For every revolution, the minute hand runs the angles between $0°$ and $360°$, so it has the same period as the standard sine and cosine functions, which is $360°$. So the parameter b is equal to 1, and the equations of the minute hand's motion are

$$x(t) = 2\sin t$$

$$y(t) = 2\cos t$$

Now find the set of all possible values of time t when $x(t) = 1$. First we need to find the set of all possible values of t such that

$$2\sin t = 1$$

$$\sin t = 0.5$$

Recall that for any k in the interval $(-1,1)$ the equation $\sin t = k$ has exactly two solutions, t_1 and t_2, within the period $[0,360°)$. Also, since the sine function has a period of $360°$, we can get other solutions by adding multiples of $360°$.

In the unit circle, the sine of an angle t is equal to the y-coordinate of the point P where the terminal side of t intersects the unit circle. Sketch the unit circle and the line $y = 0.5$. The points of their intersections P_1 and P_2 have the y-coordinate of 0.5, and so the corresponding angles are the solutions of the equation.

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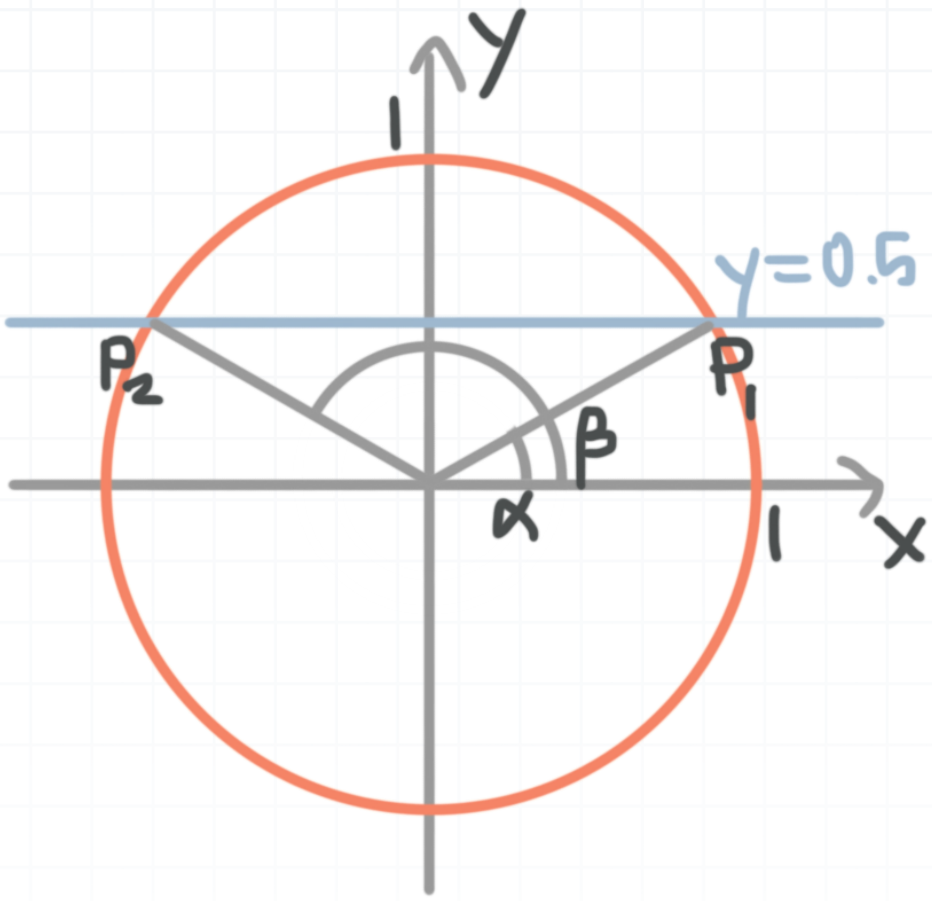

We know that

$$\sin 30^\circ = 0.5$$

The second angle we're interested in is

$$180^\circ - 30^\circ = 150^\circ$$

Therefore $t_1 = 30^\circ$ and $t_2 = 150^\circ$. Extend the solution to other periods by adding the multiples of 360°.

$t_1 = 30^\circ + 360^\circ n$, where n is any integer

$t_2 = 150^\circ + 360^\circ m$, where m is any integer

Since one minute in the clock dial corresponds to $360^\circ/60 = 6^\circ$, the time T is

$$T = \frac{t}{6^\circ}$$

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So

$$T_1 = \frac{30^\circ + 360^\circ n}{6^\circ}$$

$T_1 = 5 + 60n$, where n is any integer

and

$$T_2 = \frac{150^\circ + 360^\circ m}{6^\circ}$$

$T_2 = 25 + 60m$, where m is any integer

So during the period of one hour, the answers are 5 minutes and 25 minutes.

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GRAPHS OF INVERSE TRIG RELATIONS

■ 1. Sketch the graph of the inverse relation for the function.

$$f(x) = 2\sin(x - \pi) + 1$$

Solution:

Since the graphs of a function and its inverse are reflections over the line $y = x$, we can build the graph of the function, and then just reflect it over the line $y = x$.

Let's sketch the graph of $f(x) = 2\sin(x - \pi) + 1$ first. We can get it from the graph of the standard sine function using the following sequence of transformations:

1) The initial function is $f_1(x) = \sin x$, and the sketch of its graph is

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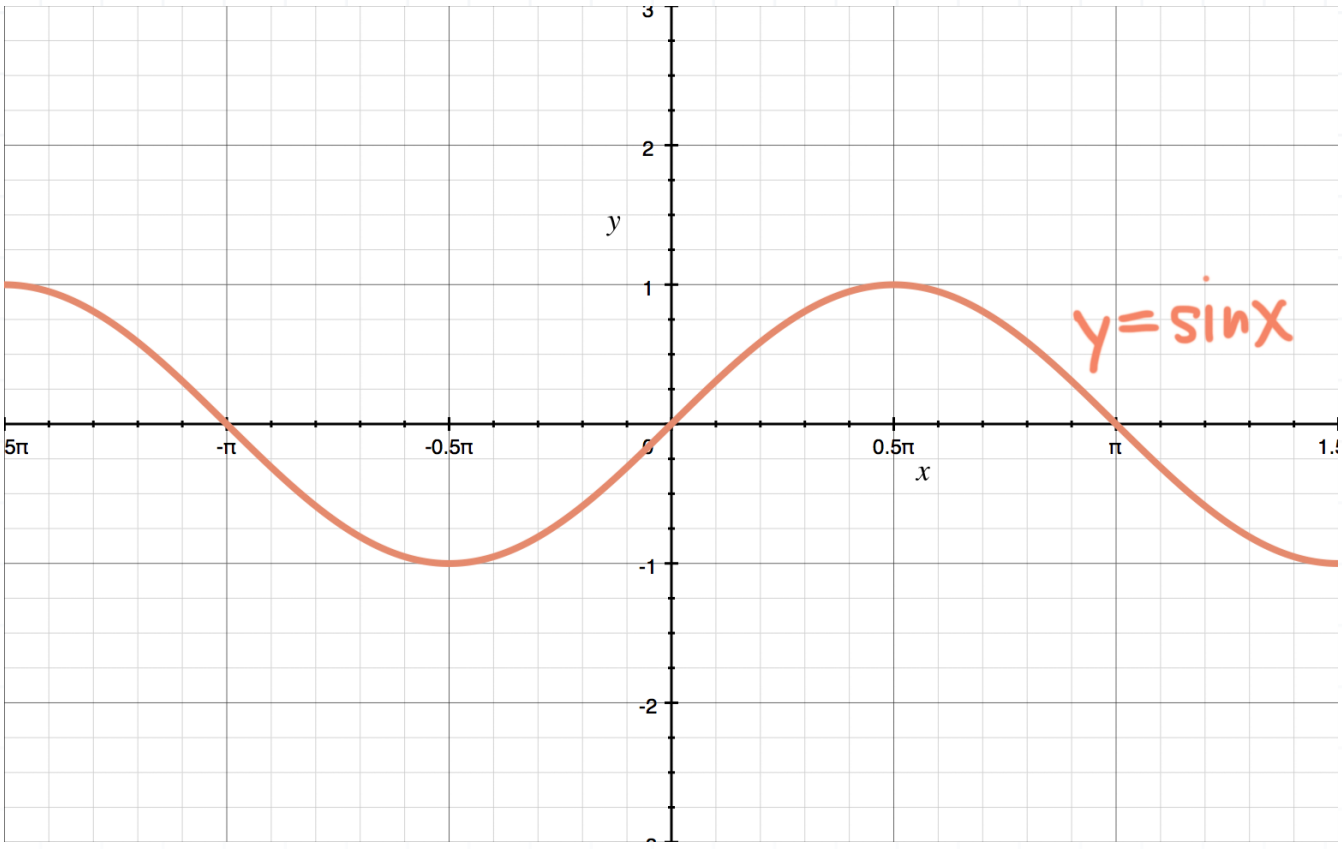
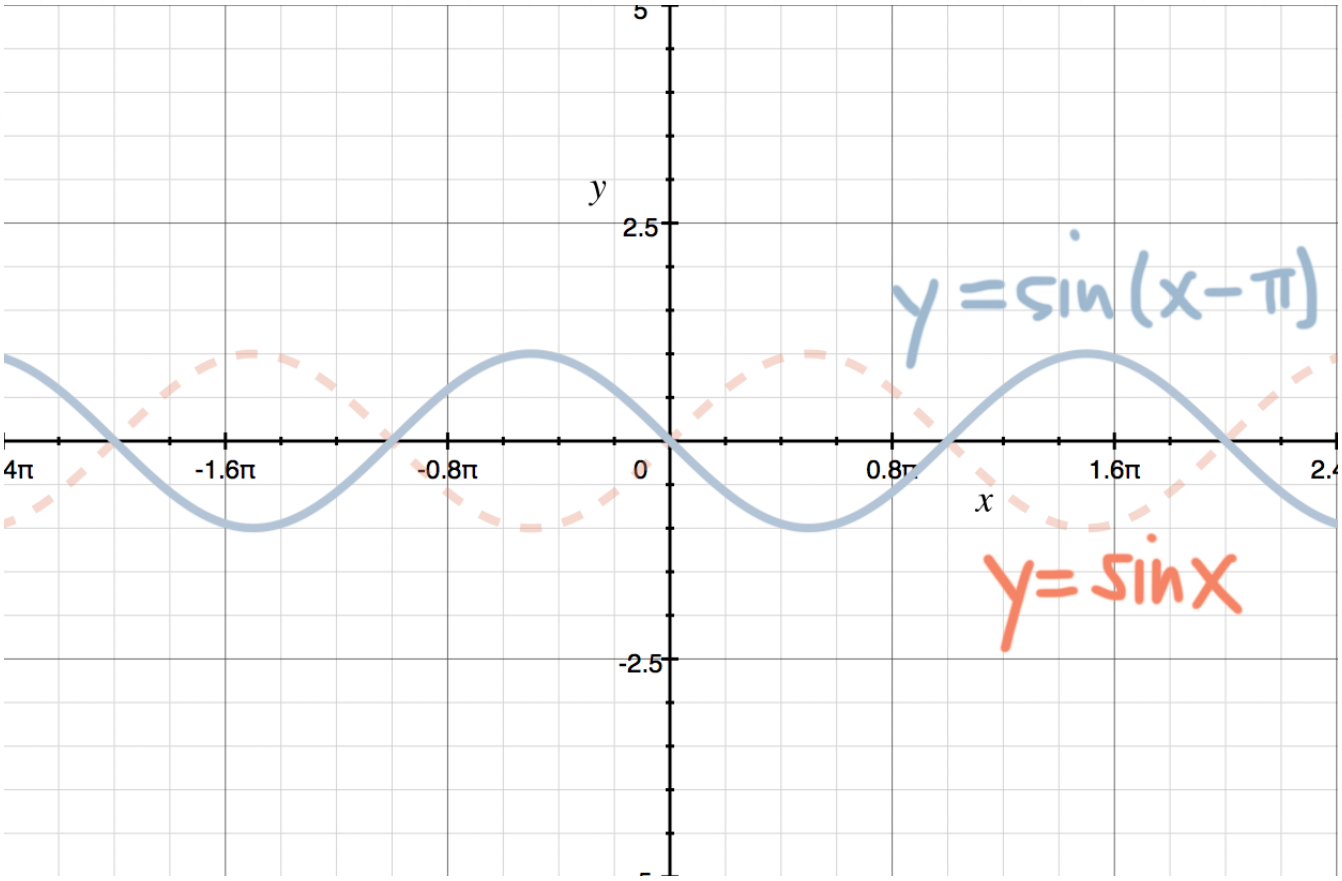


2) The horizontal shift to the right by π gives $f_2(x) = f_1(x - \pi) = \sin(x - \pi)$, and the sketch of its graph is



3) The vertical stretch by a factor of 2 gives $f_3(x) = 2f_2(x) = 2\sin(x - \pi)$, and the sketch of its graph is

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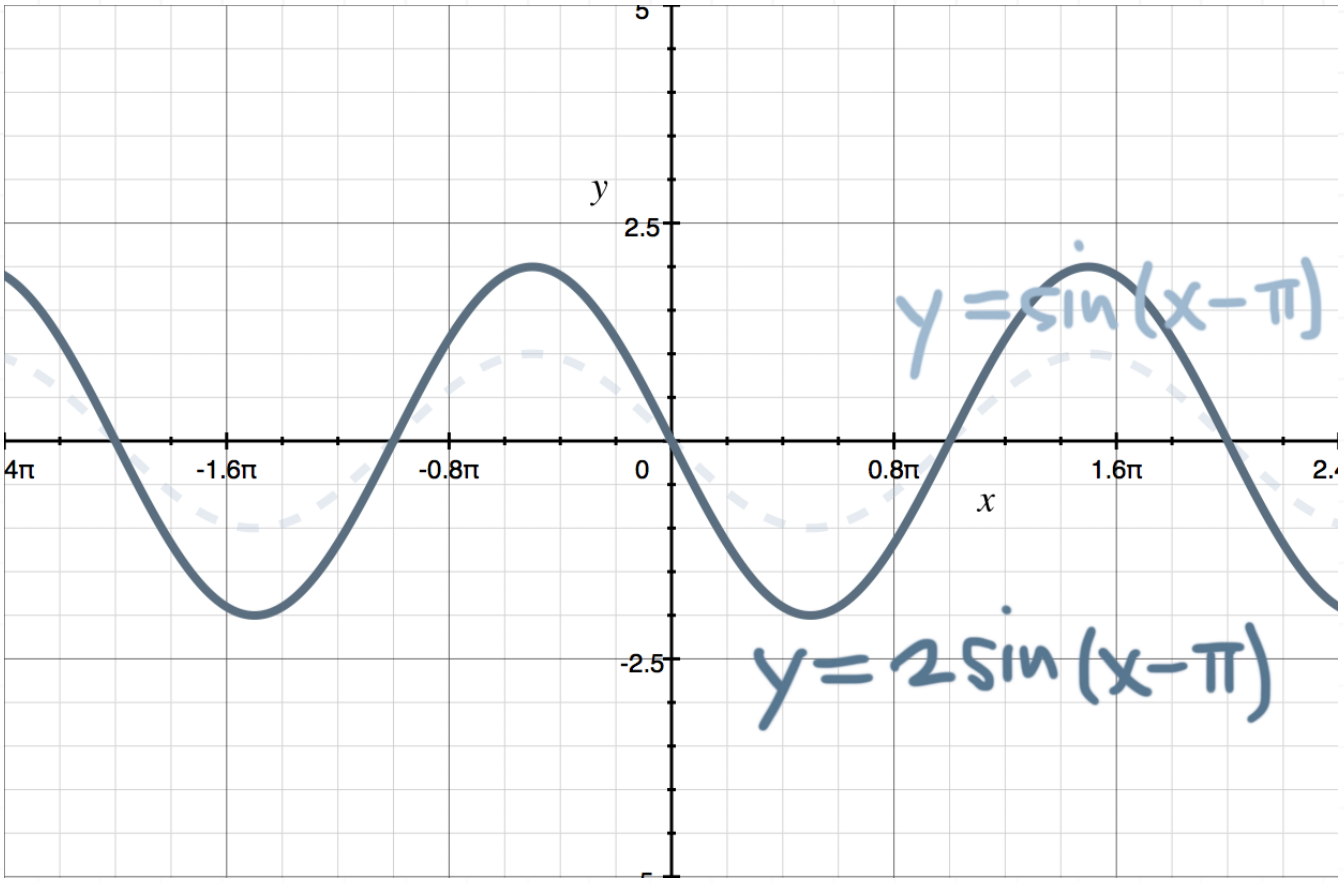
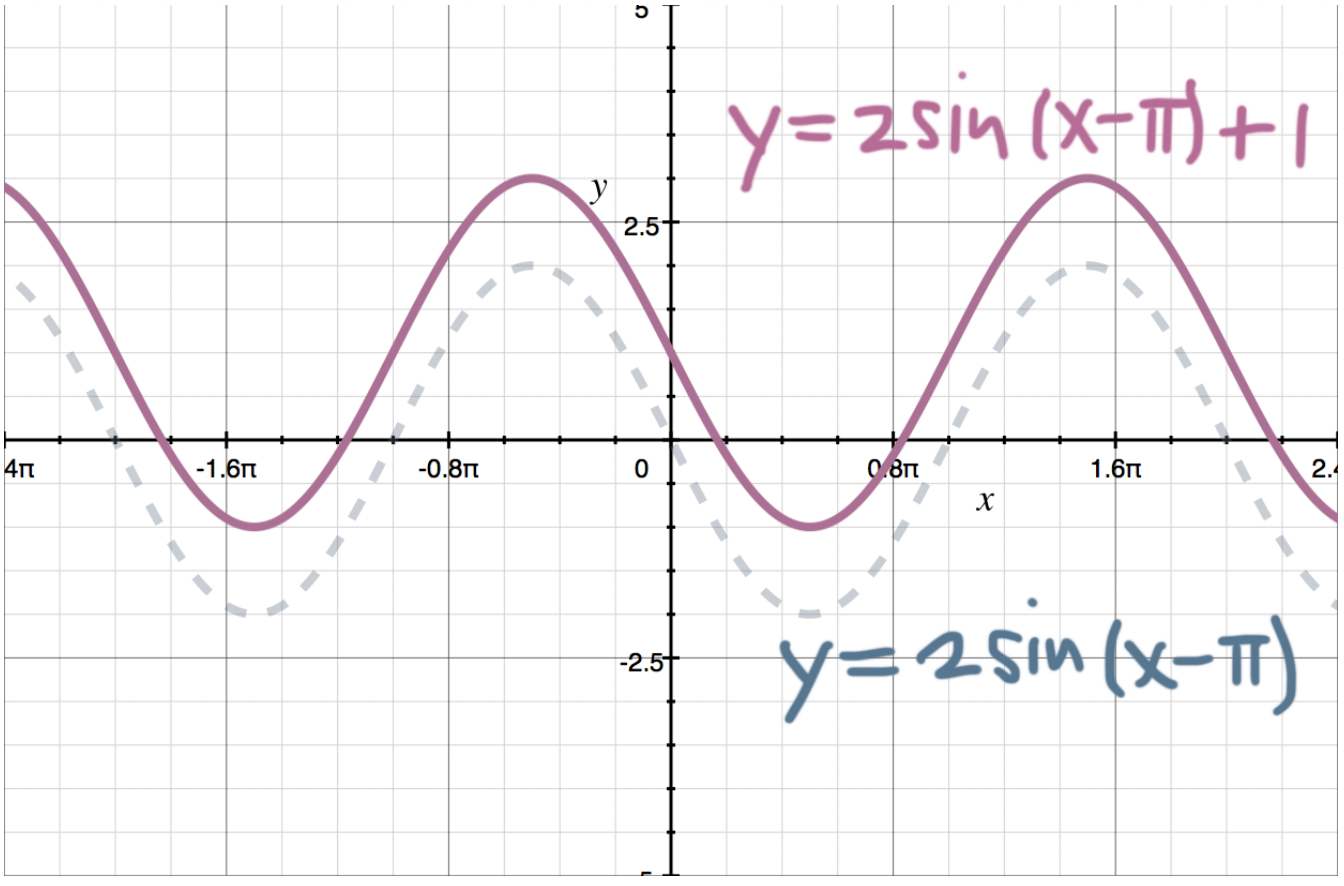


4) The vertical shift up by 1 gives $f_4(x) = f_3(x) + 1 = 2\sin(x - \pi) + 1$, and the sketch of its graph is



As we can see, $f_4(x) = f(x)$. The sketch of just $f_4(x)$ is

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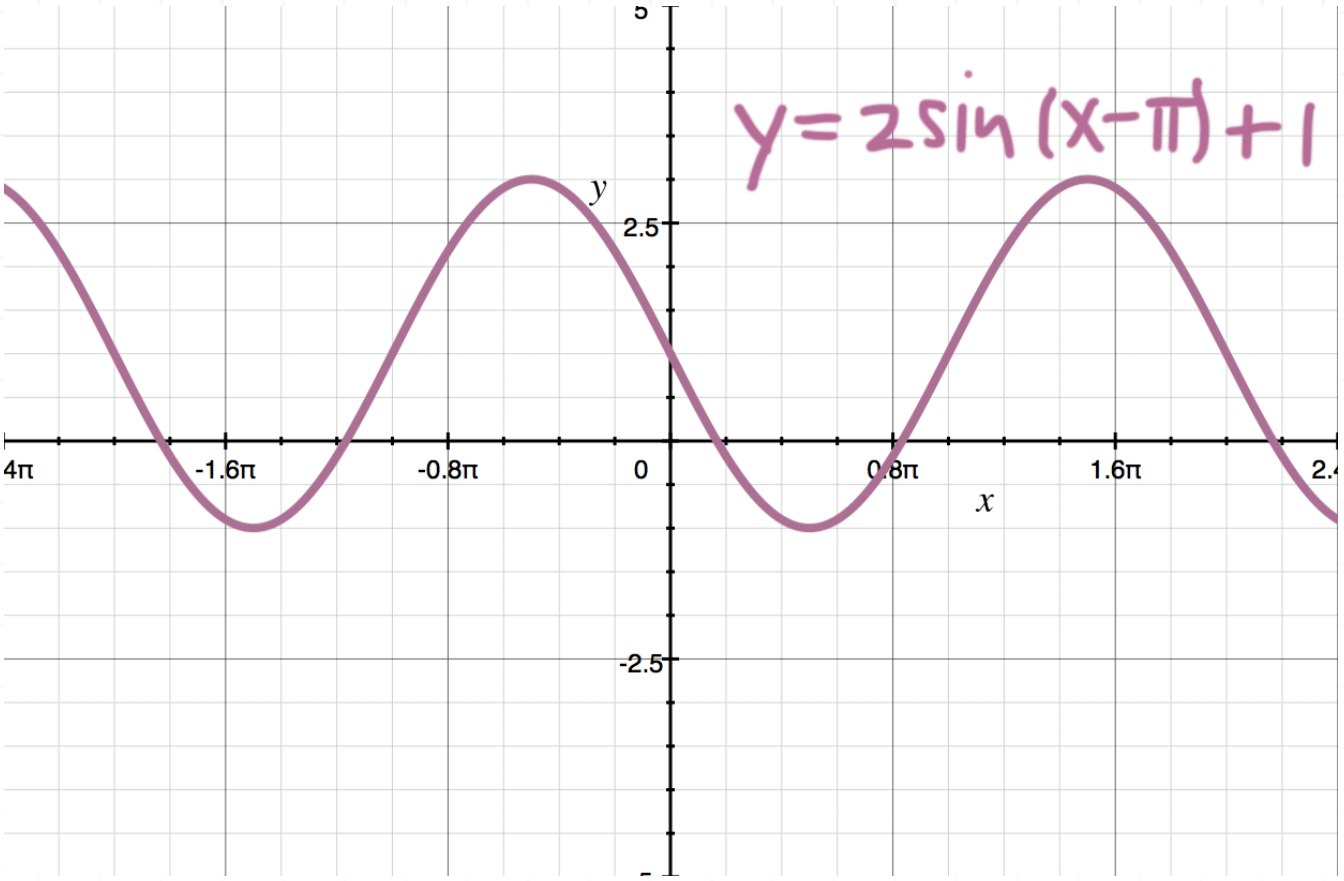

Now, we need to reflect the graph over the line $y = x$. We can calculate several points on the graph, and then swap their coordinates to get the corresponding points on the inverse graph.

We know these are points on the original graph:

$$\left(-\frac{\pi}{2}, 3\right), \left(\frac{\pi}{2}, -1\right), \text{ and } \left(\frac{3\pi}{2}, 3\right)$$

Swap their x- and y-coordinates to get these points on the inverse:

$$\left(3, -\frac{\pi}{2}\right), \left(-1, \frac{\pi}{2}\right), \text{ and } \left(3, \frac{3\pi}{2}\right)$$

Sketch these points and then join them to get the graph of the inverse relation.

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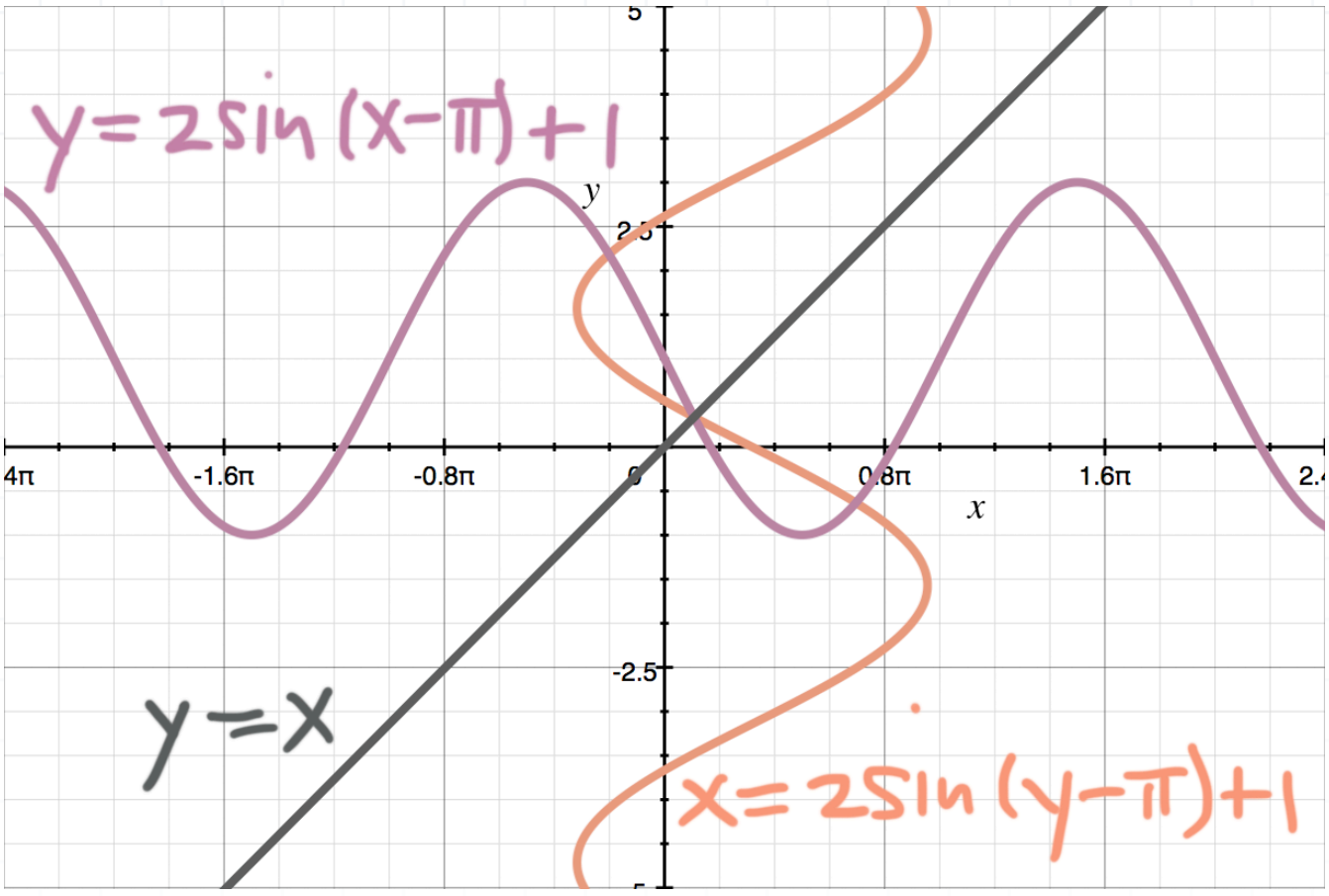
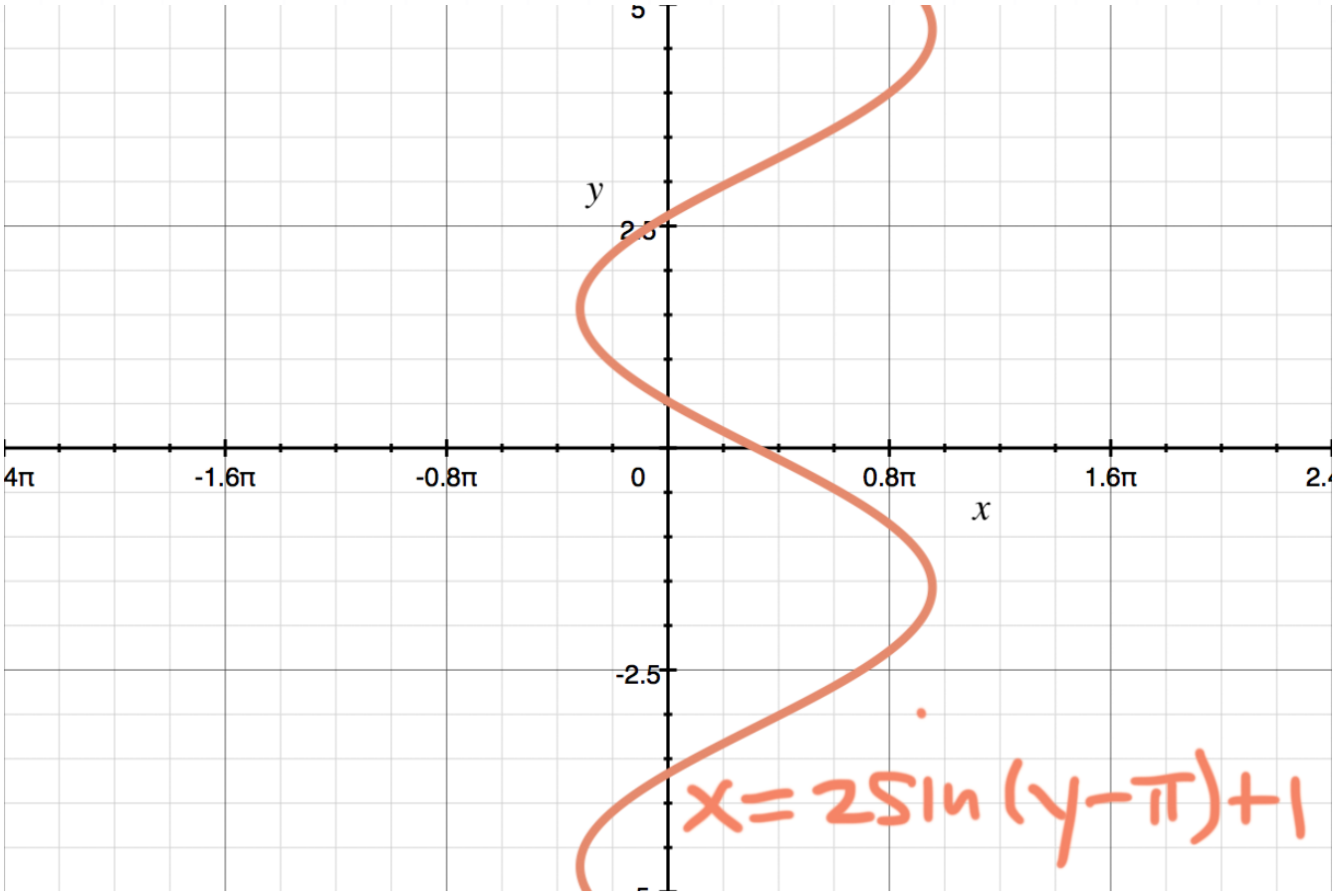


So the sketch of the inverse is



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2. Sketch the graph of the inverse relation for the function.

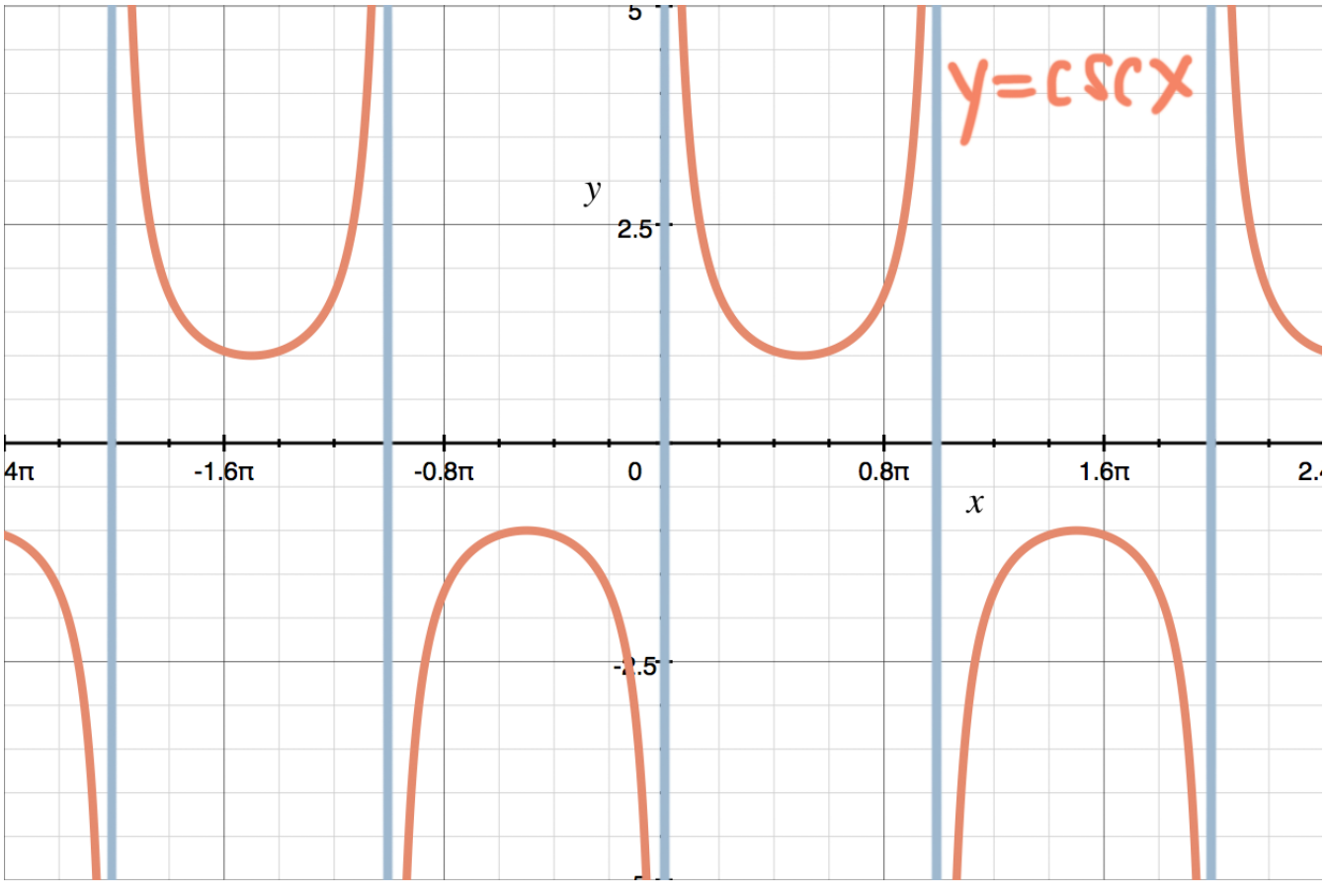
$$f(x) = \frac{1}{2}\csc\left(x + \frac{\pi}{2}\right) - 1.5$$

Solution:

Since the graphs of a function and its inverse are reflections over the line $y = x$, we can build the graph of the function, and then just reflect it over the line $y = x$.

Let's sketch the graph of $f(x)$ first. We can get it from the graph of the standard cosecant function using the following sequence of transformations:

1) The initial function is $f_1(x) = \csc x$, and the sketch of its graph is



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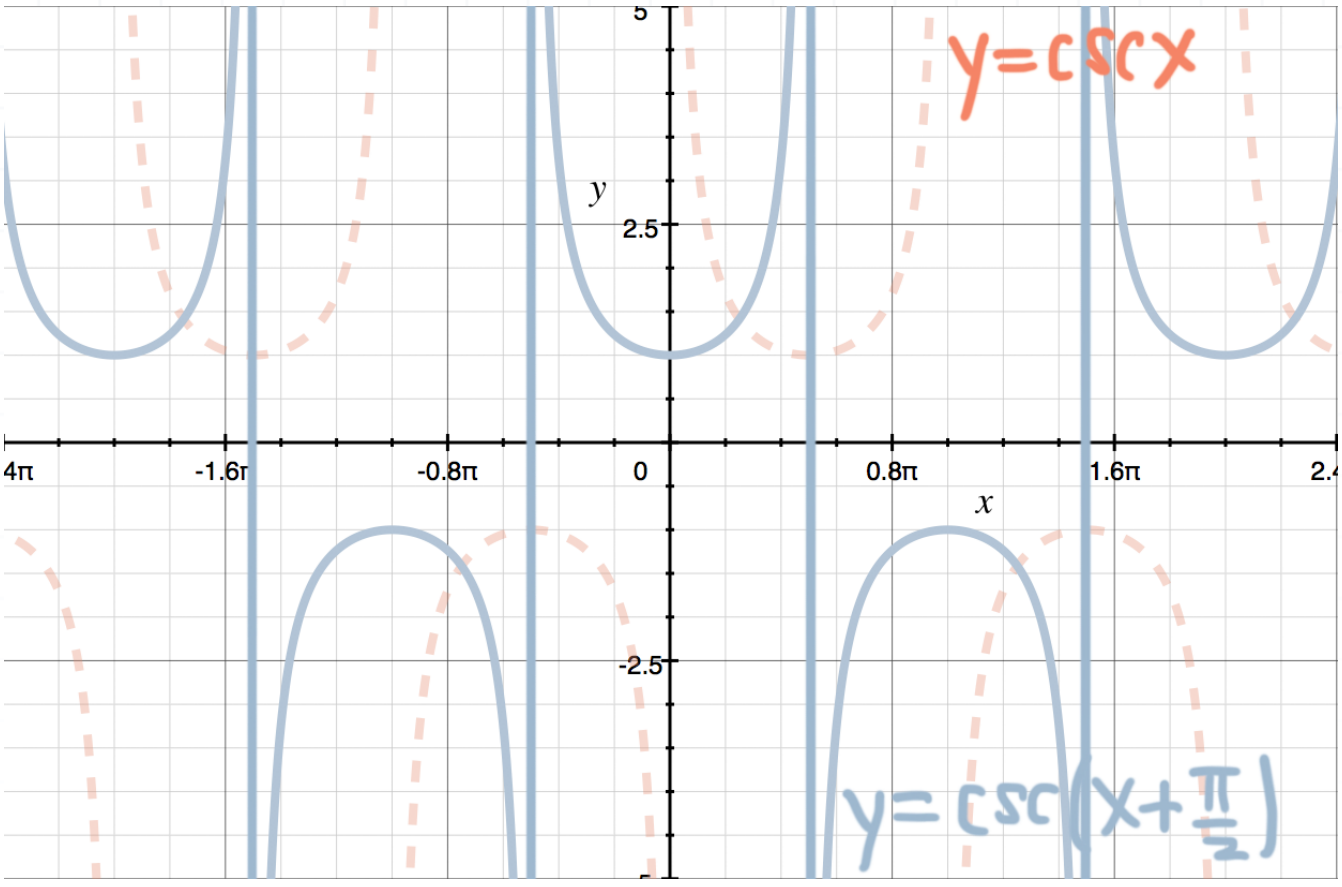
2) The horizontal shift to the left by $\pi/2$ gives

$$f_2(x) = f_1\left(x + \frac{\pi}{2}\right) = \csc\left(x + \frac{\pi}{2}\right)$$

And the sketch of its graph is



Therefore, after the shift, the first period of $(0,2\pi)$ moved to the left into the period $(-\pi/2,3\pi/2)$. The new asymptotes are $x = -\pi/2$, $x = \pi/2$, and $x = 3\pi/2$.

3) The vertical compression by a factor of $1/2$ gives

$$f_3(x) = \frac{1}{2} f_2(x) = \frac{1}{2}\csc\left(x + \frac{\pi}{2}\right)$$

And the sketch of its graph is

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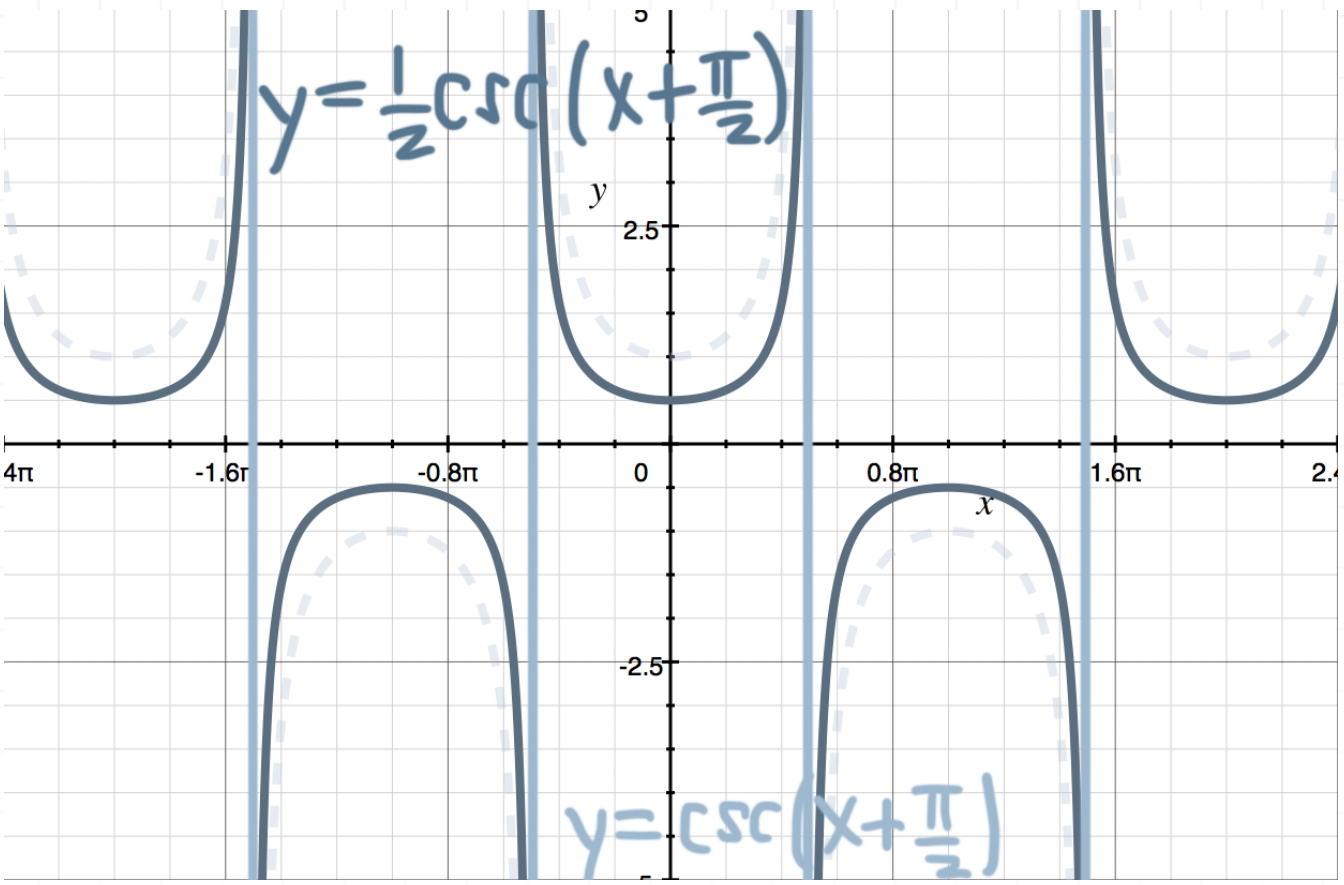

The period and asymptotes aren't affected by any vertical transformation.

4) The vertical shift down by 1.5 gives

$$f_4(x) = f_3(x) - 1.5 = \frac{1}{2}\csc\left(x + \frac{\pi}{2}\right) - 1.5$$

And the sketch of its graph is

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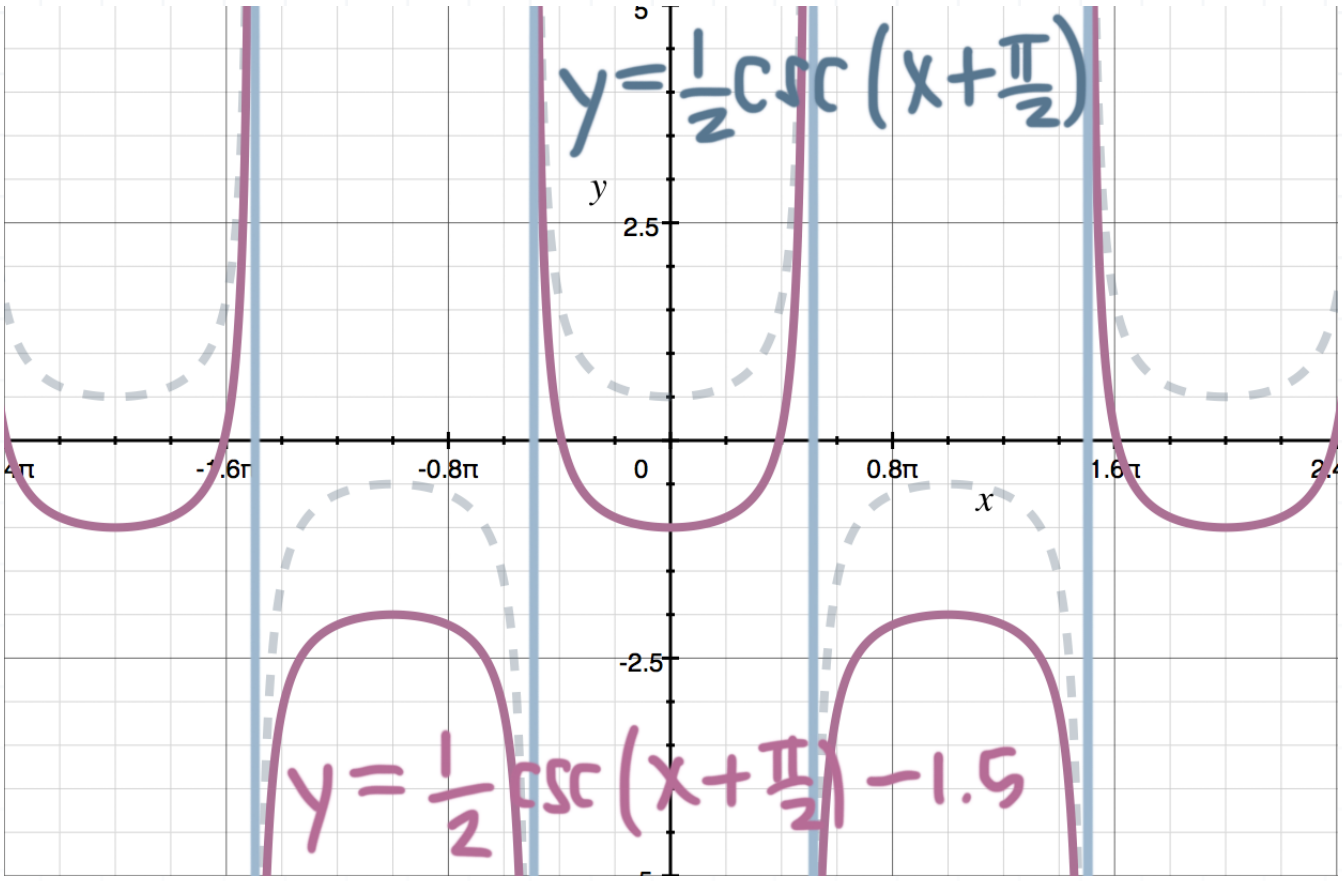
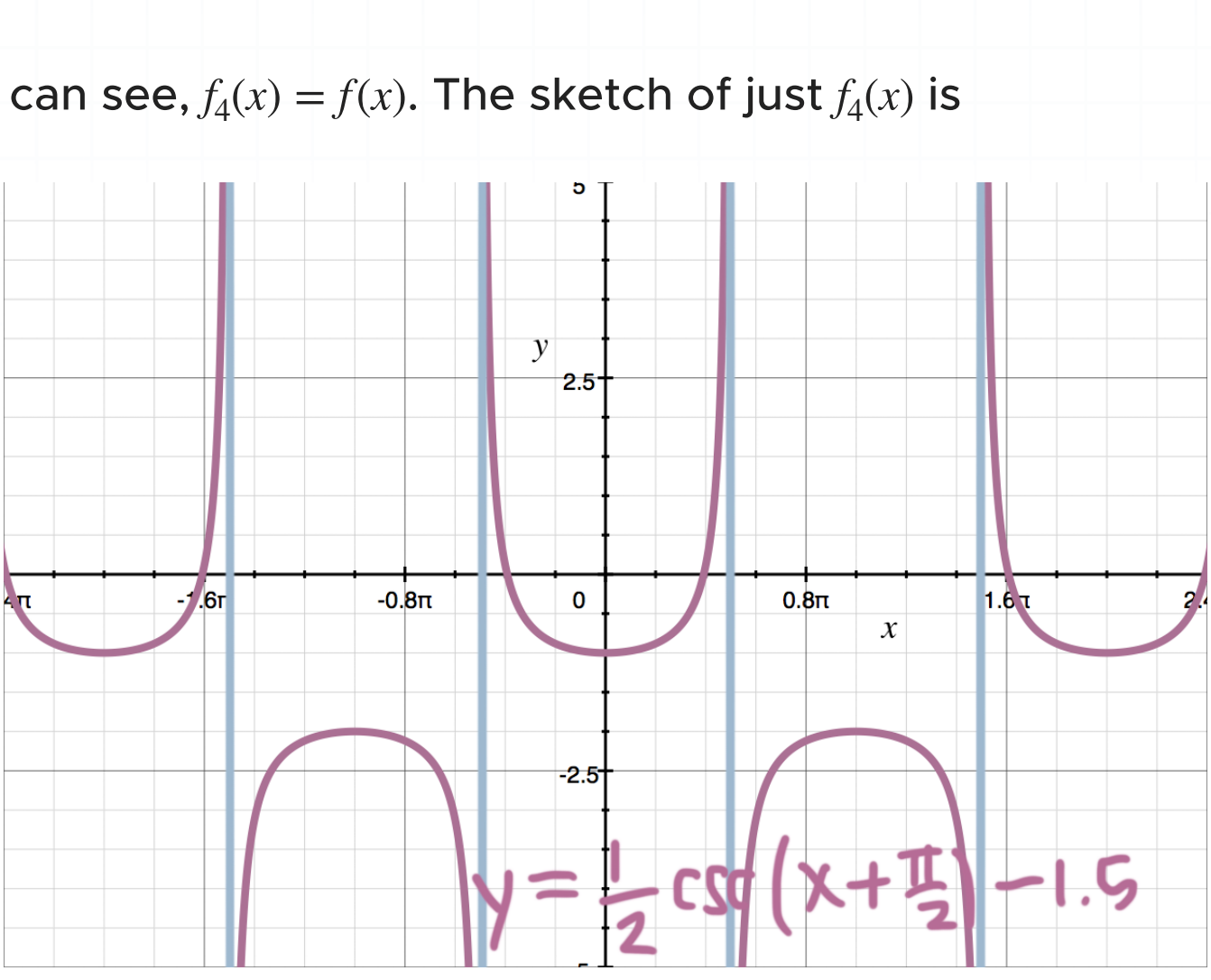


As we can see, $f_4(x) = f(x)$. The sketch of just $f_4(x)$ is



Now, we need to reflect the graph over the line $y = x$. Let's calculate new asymptotes and several points on the inverse relation graph.

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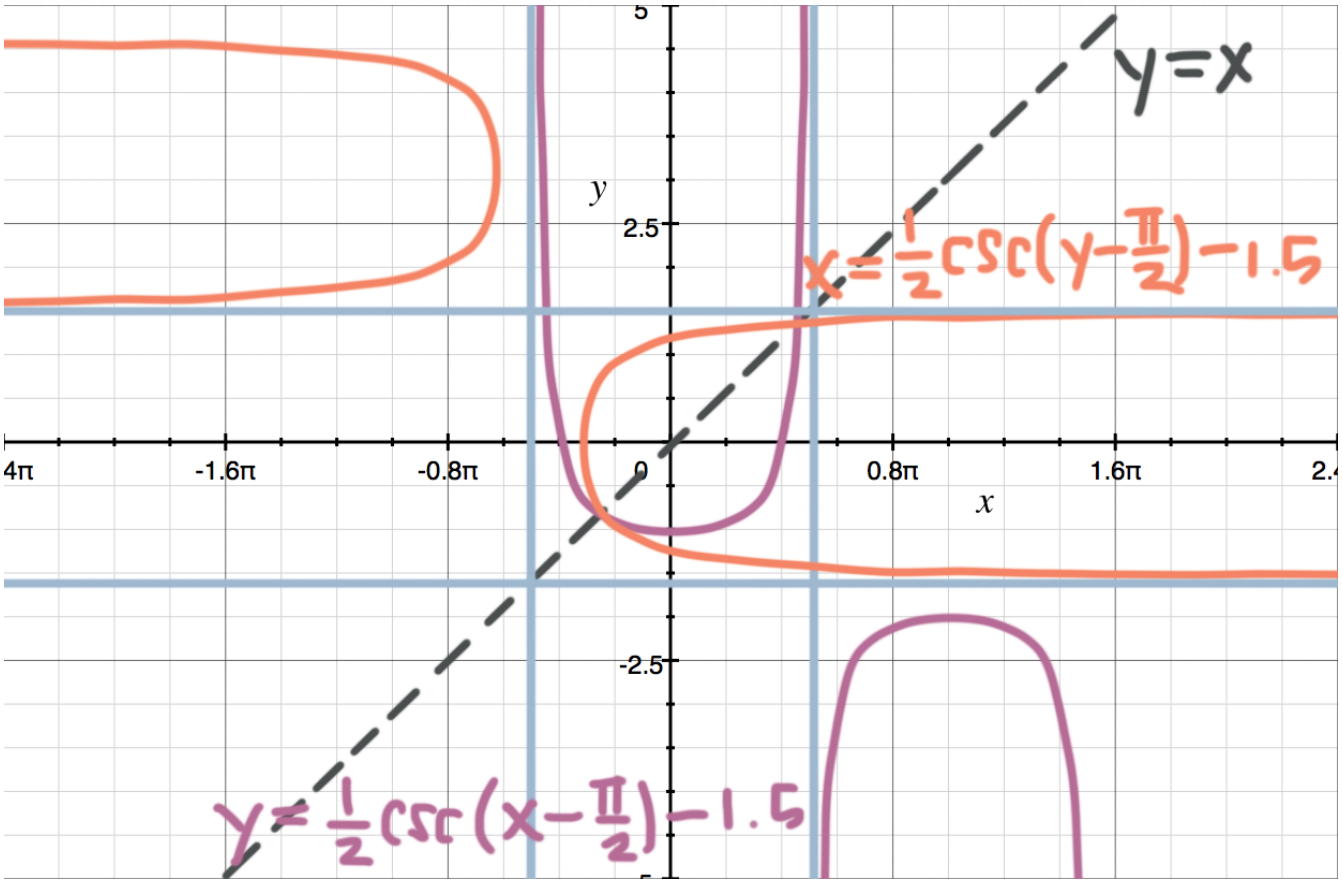
Consider the first period $(-\pi/2,3\pi/2)$, with asymptotes $x = -\pi/2$, $x = \pi/2$, and $x = 3\pi/2$.

The minimum point of the left-hand branch is $(0, -1)$, and the maximum point of the right-hand branch is $(\pi, -2)$.

To get reflected points on the inverse relation graph, we just need to swap the places of the x- and y-coordinates of these points. So the new points have the coordinates $(-1,0)$ and $(-2,\pi)$.

We also need to swap the coordinates of the period and asymptotes. The first period of the reflected graph is $y \in (-\pi/2,3\pi/2)$, with asymptotes $y = -\pi/2$, $y = \pi/2$, and $y = 3\pi/2$.

The graph of the first period of the inverse relation is



We can get the full graph by duplicating the first period.

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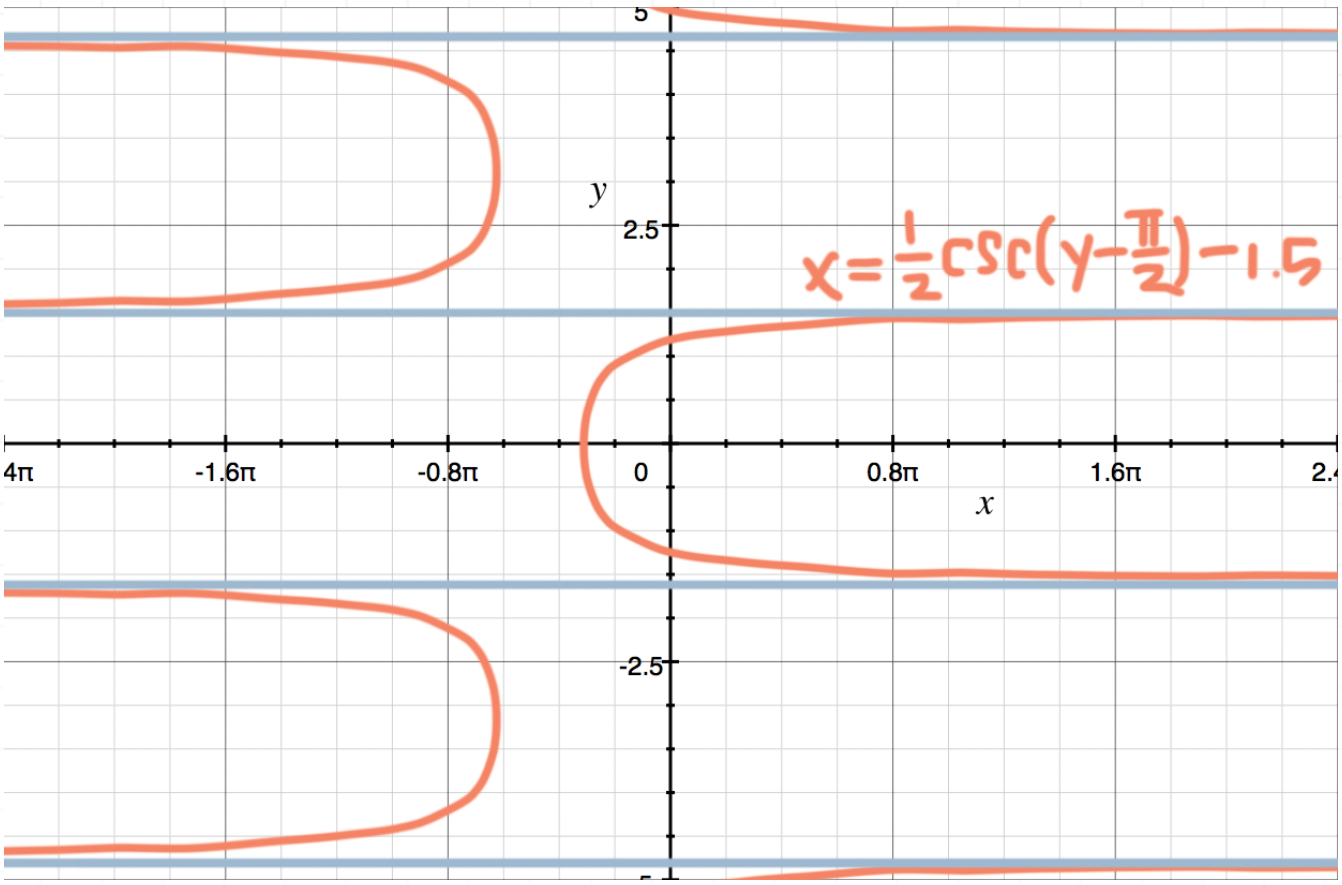

■ 3. Sketch the graph of the inverse relation for the function.

$$f(x)=\frac{1}{\sqrt{3}}\tan\left(\frac{\pi}{3}x-\frac{\pi}{3}\right)+2$$

Solution:

Since the graphs of a function and its inverse are reflections over the line $y=x$, we can build the graph of the function, and then just reflect it over the line $y=x$.

Let's sketch the graph of $f(x)$ first. Factor out $\pi/3$ inside the tangent function.

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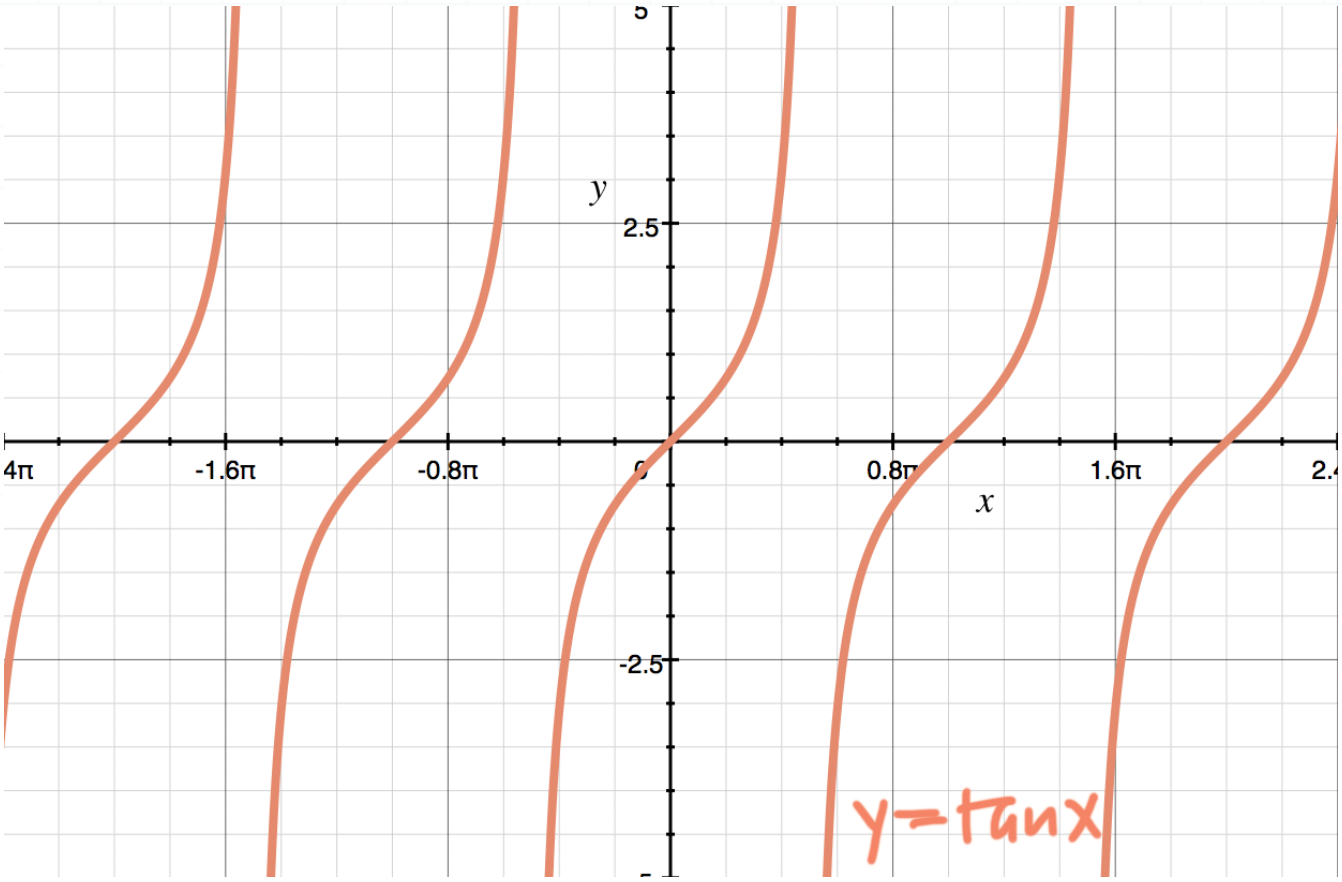
$$f(x) = \frac{1}{\sqrt{3}} \tan\left(\frac{\pi}{3}(x-1)\right) + 2$$

We can get it from the graph of the standard tangent function using the following sequence of transformations:

1) The initial function is $f_1(x) = \tan x$, and the sketch of its graph is



2) The horizontal stretch by a factor of $\pi/3$ gives

$$f_2(x) = f_1\left(\frac{\pi}{3}x\right) = \tan\left(\frac{\pi}{3}x\right)$$

And a sketch of its graph is

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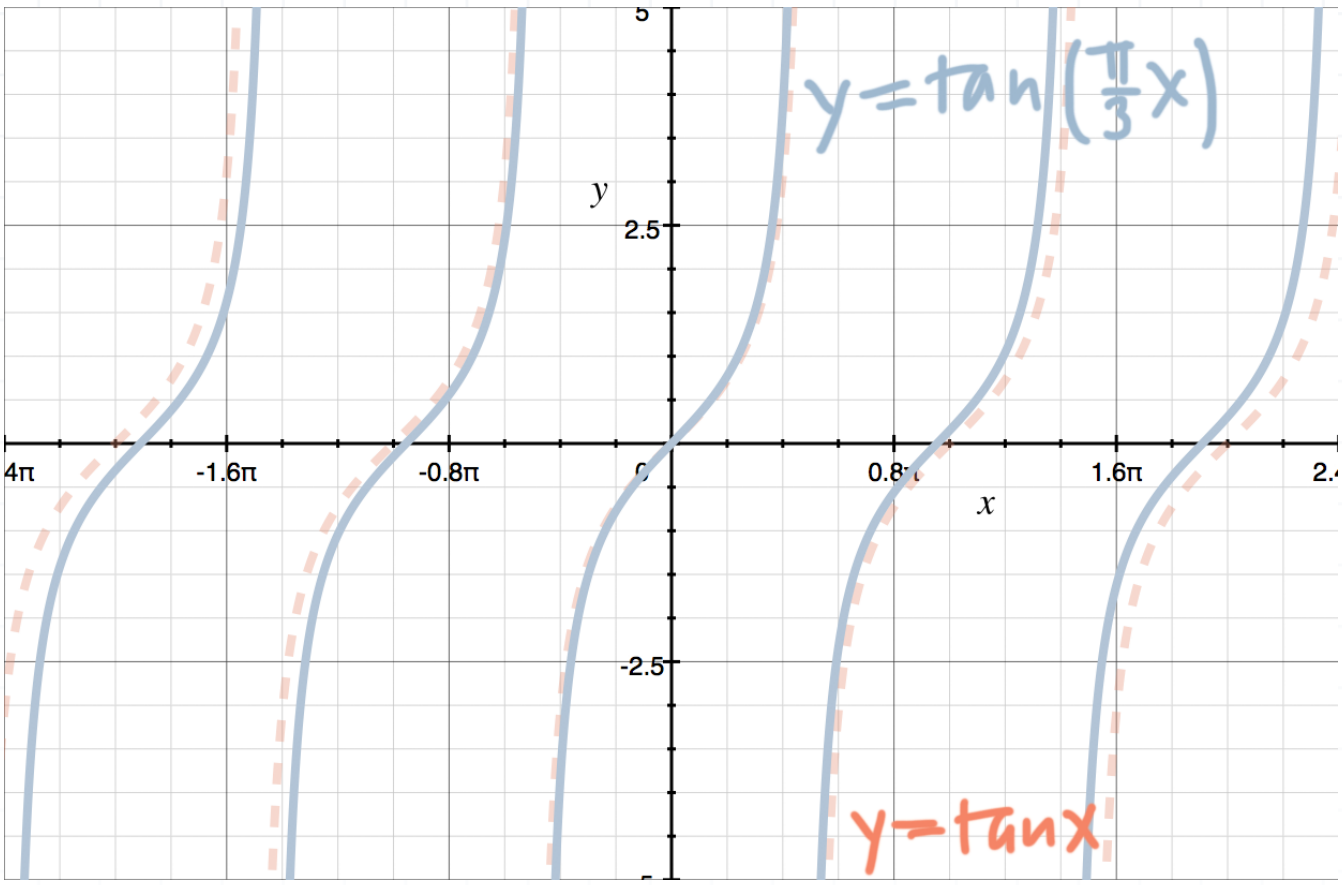

We know that if the period of the function $g(x)$ is π, then the period of $g(bx)$ is

$$\text{period} = \frac{\pi}{b} = \frac{\pi}{\frac{\pi}{3}} = 3$$

So after the stretching, the first period of $(-\pi/2,\pi/2)$ transformed into the period $(-1.5,1.5)$. The new asymptotes are $x = -1.5$ and $x = 1.5$.

3) The horizontal shift to the right by 1 gives

$$f_3(x) = f_2(x-1) = \tan\left(\frac{\pi}{3}(x-1)\right)$$

After the shift, the first period of $(-1.5,1.5)$ moved to the right into the period $(-0.5,2.5)$. The new asymptotes are $x = -0.5$ and $x = 2.5$.

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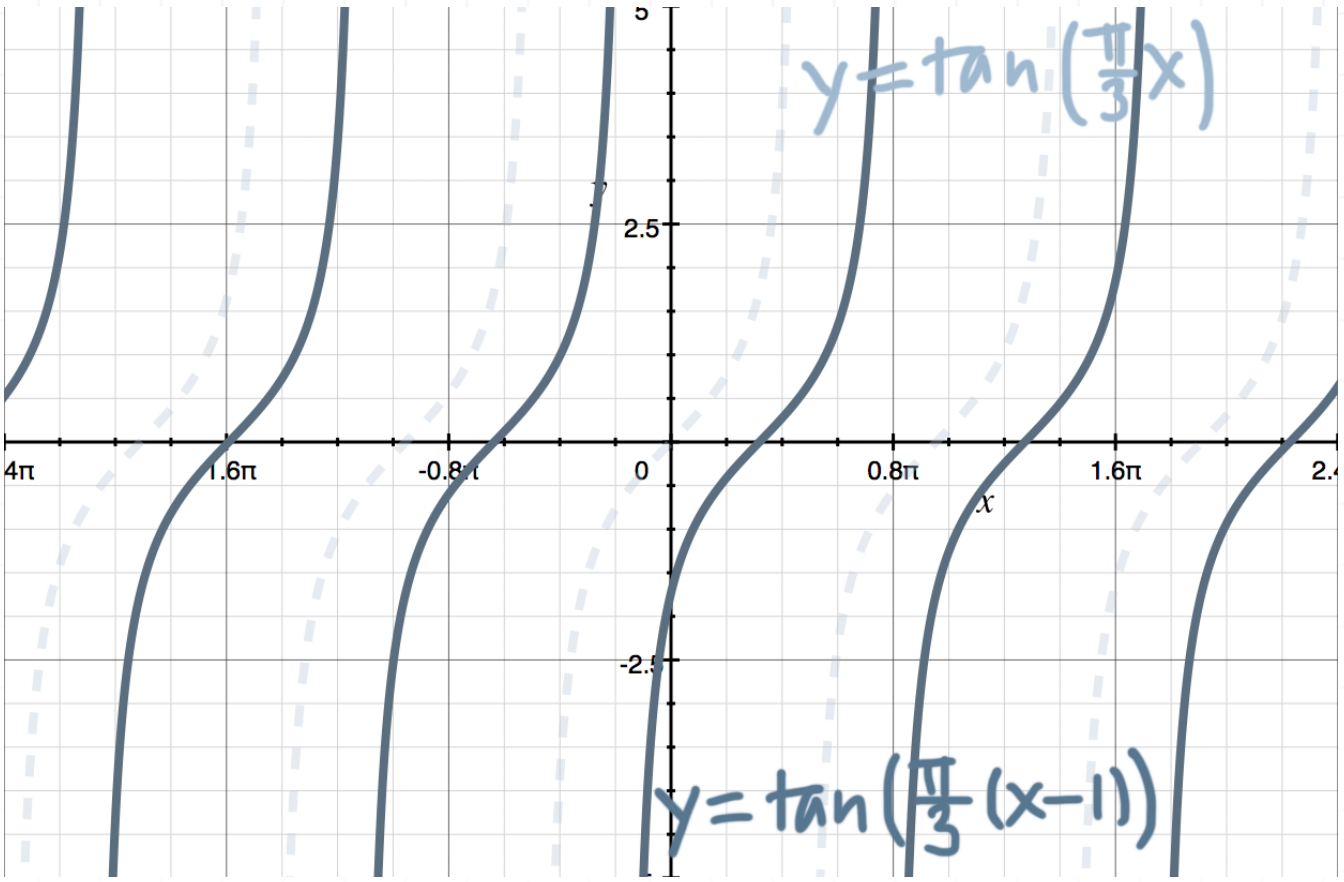

4) The vertical compression by a factor of $1/\sqrt{3}$ gives

$$f_4(x) = \frac{1}{\sqrt{3}} f_3(x) = \frac{1}{\sqrt{3}} \tan\left(\frac{\pi}{3}(x-1)\right)$$

The period and asymptotes aren't affected by a vertical transformation.

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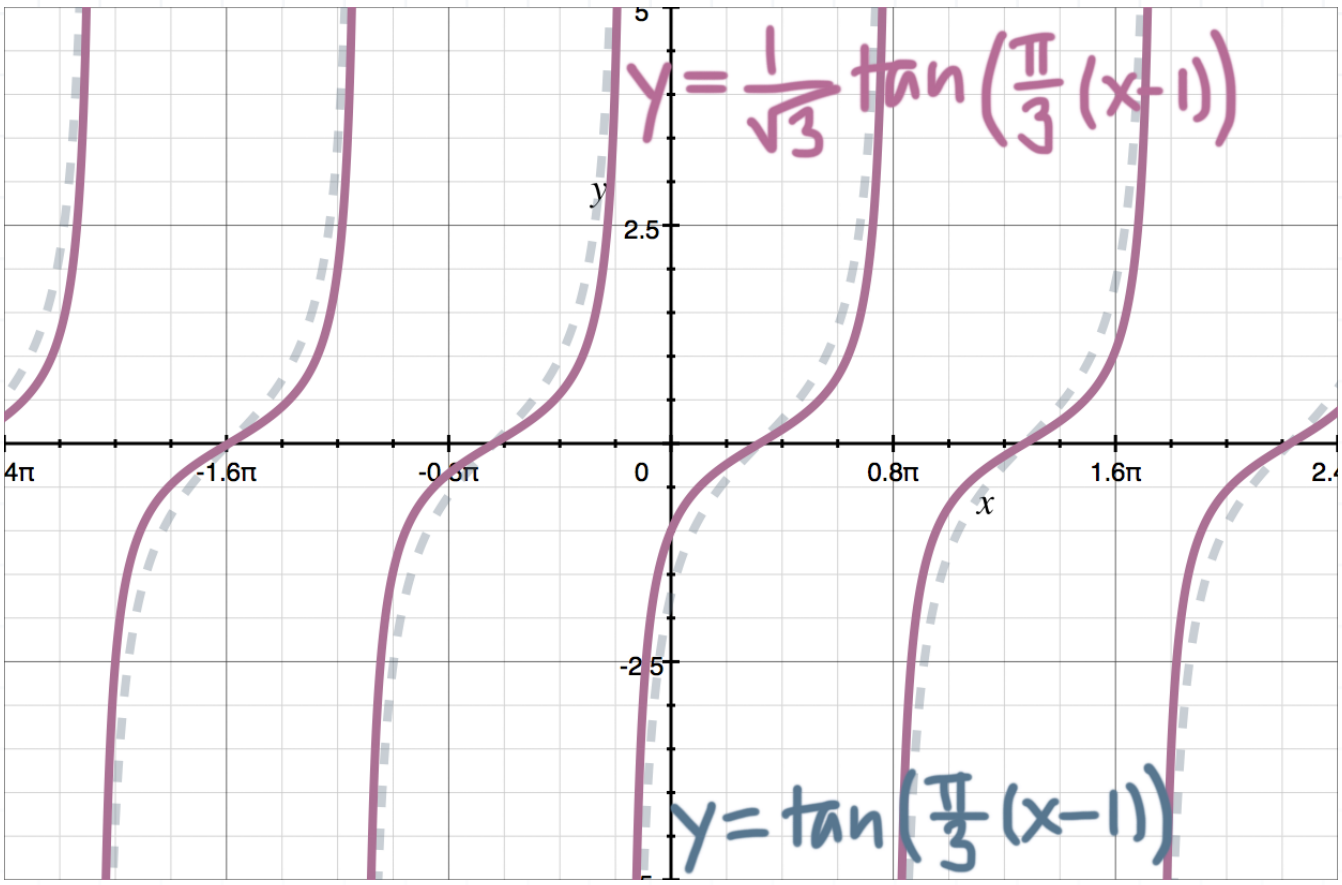

5) The vertical shift up by 2 gives

$$f_5(x) = f_4(x) + 2 = \frac{1}{\sqrt{3}} \tan\left(\frac{\pi}{3}(x-1)\right) + 2$$

And the sketch of its graph is

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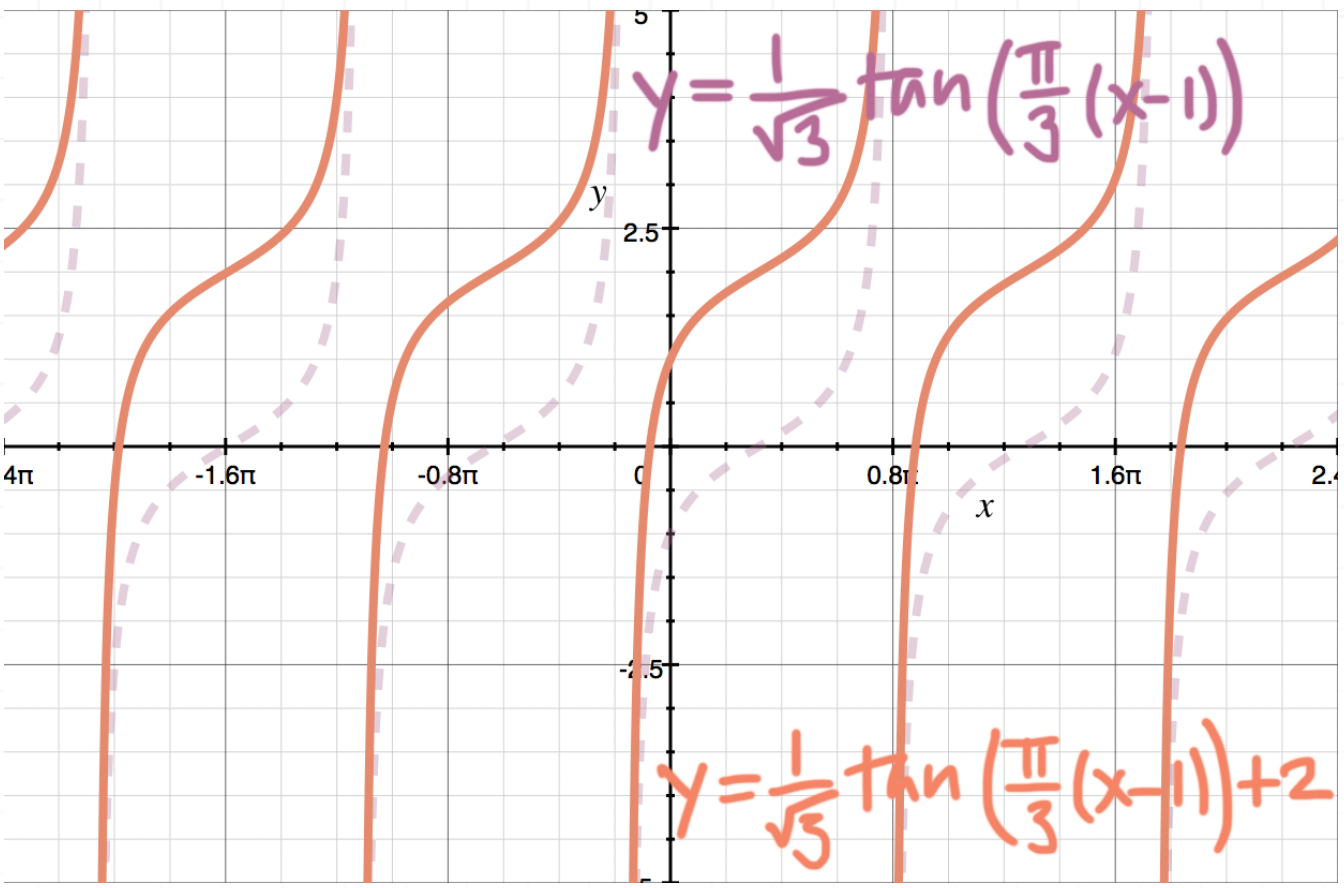
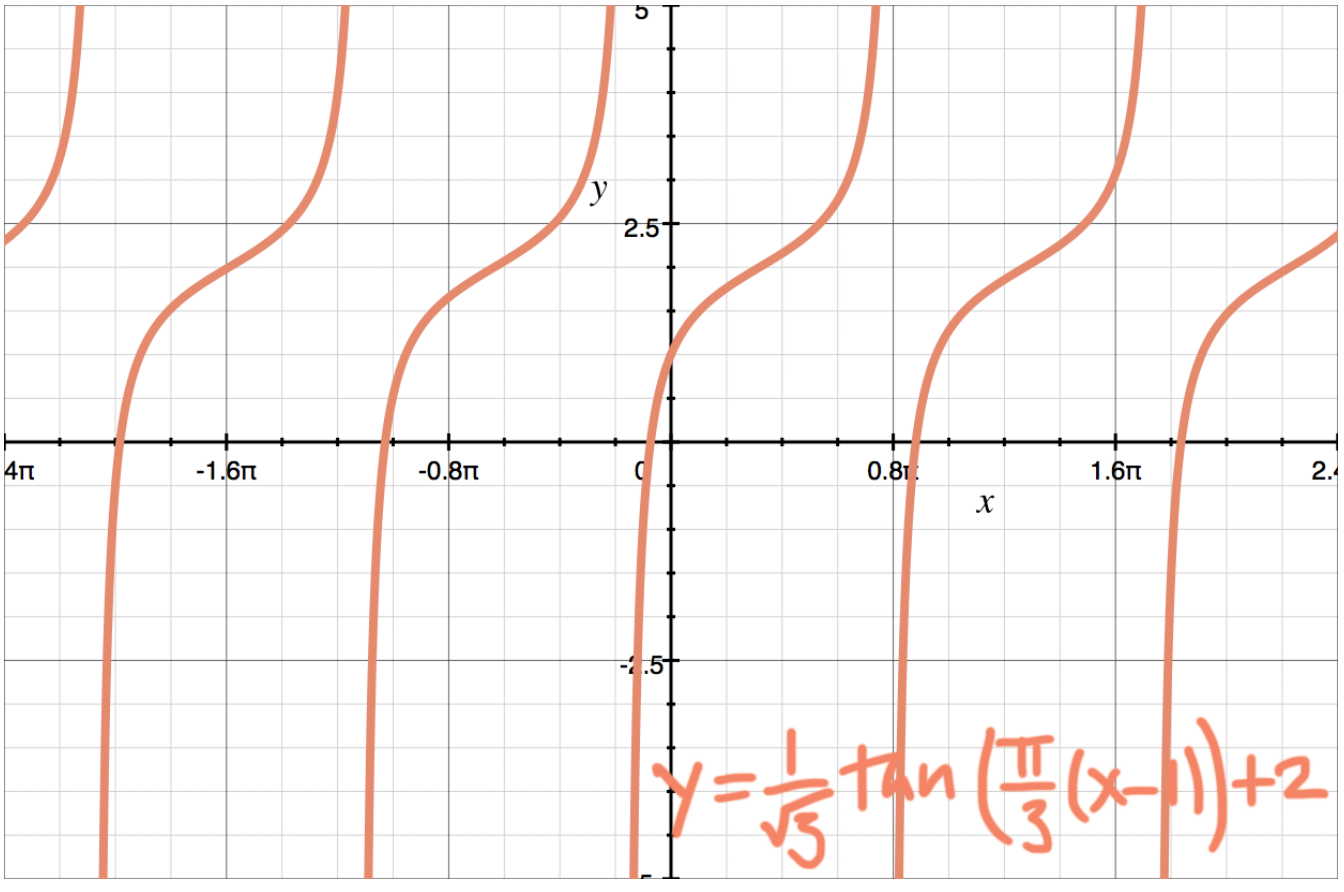


As we can see, $f_5(x) = f(x)$. The graph of just $f_5(x)$ is



Now we need to reflect the graph over the line $y = x$. Let's calculate new asymptotes and several points on the inverse relation graph.

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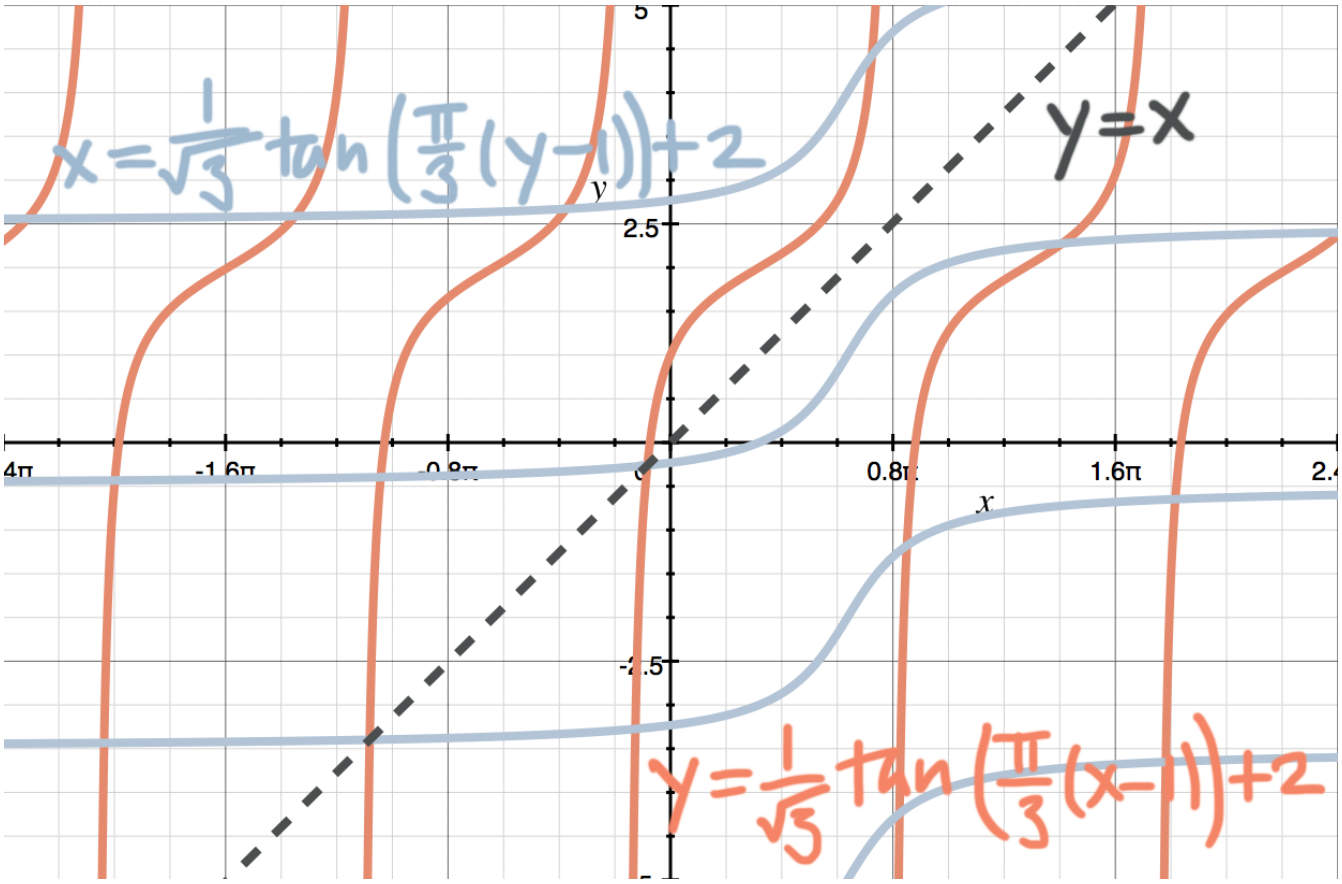
Let's consider the first period $x \in (-0.5,2.5)$, with asymptotes $x = -0.5$ and $x = 2.5$.

Three points with integer coordinates on the graph of the original function are $(0,1)$, $(1,2)$, and $(2,3)$.

To get the reflected points on the inverse relation graph, we just need to swap the x- and y- coordinates of these points. So the new points have the coordinates $(1,0)$, $(2,1)$, and $(3,2)$.

We also need to swap coordinates for the period and asymptotes. The first period of the reflected graph is $y \in (-0.5,2.5)$, with asymptotes $y = -0.5$ and $y = 2.5$.

The graph of the first period of the inverse relation is



We can get the full graph by duplicating the first period.

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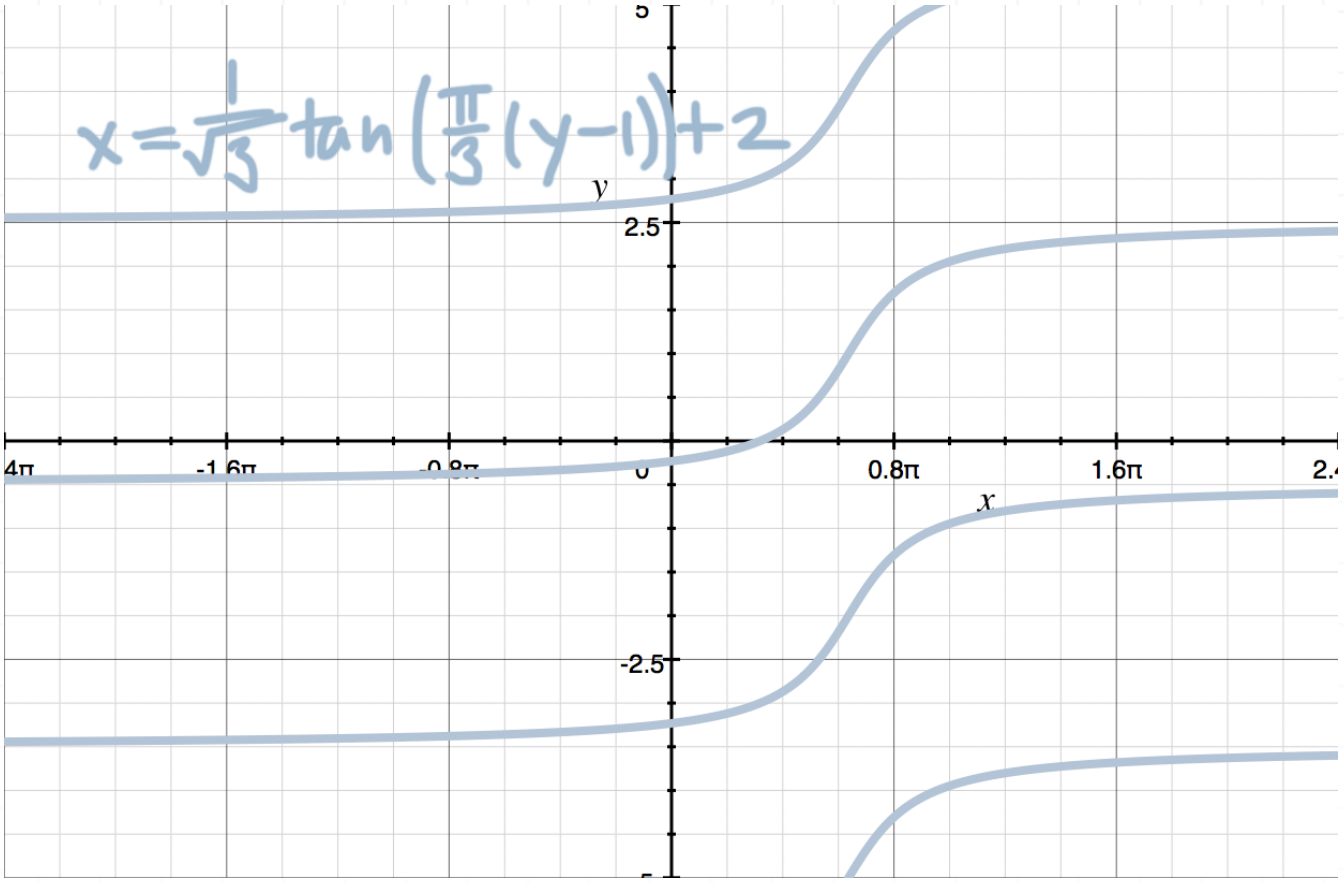

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FINDING THE VALUE OF AN INVERSE TRIG FUNCTION

■ 1. Prove the identities or give a counterexample to show that they're not true.

$$\tan^{-1} a = \frac{\sin^{-1} a}{\cos^{-1} a}$$

$$\cot^{-1} a = \frac{\cos^{-1} a}{\sin^{-1} a}$$

$$a \in [-1,1]$$

Solution:

The identities aren't correct for an arbitrary angle $a \in [-1,1]$. We can conclude that from the fact that the result of any inverse trigonometric function is an angle measured in degrees or radians. And the left side of the expression is an angle, but the right side is the quotient of two angles.

Let's take the value $a = 1/2$ as a counterexample. Since $\sin(\pi/6) = 1/2$,

$$\frac{\pi}{6} = \sin^{-1} \frac{1}{2}$$

Since $\cos(\pi/3) = 1/2$,

$$\frac{\pi}{3} = \cos^{-1} \frac{1}{2}$$

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So we can say

$$\frac{\sin^{-1}\frac{1}{2}}{\cos^{-1}\frac{1}{2}} = \frac{\frac{\pi}{6}}{\frac{\pi}{3}} = \frac{\pi}{6}\cdot\frac{3}{\pi} = \frac{1}{2}$$

But,

$$\tan^{-1}\frac{1}{2} \approx 0.46 \neq \frac{1}{2}$$

so the tangent identity is incorrect. Similarly,

$$\frac{\cos^{-1}\frac{1}{2}}{\sin^{-1}\frac{1}{2}} = 2$$

$$\cot^{-1}\frac{1}{2} \approx 1.11 \neq 2$$

■ 2. Write down the exact solution of the equation using inverse trig functions.

$$5\sin(x-4)-3=0$$

Solution:

Isolate the sine function.

$$5\sin(x-4)=3$$

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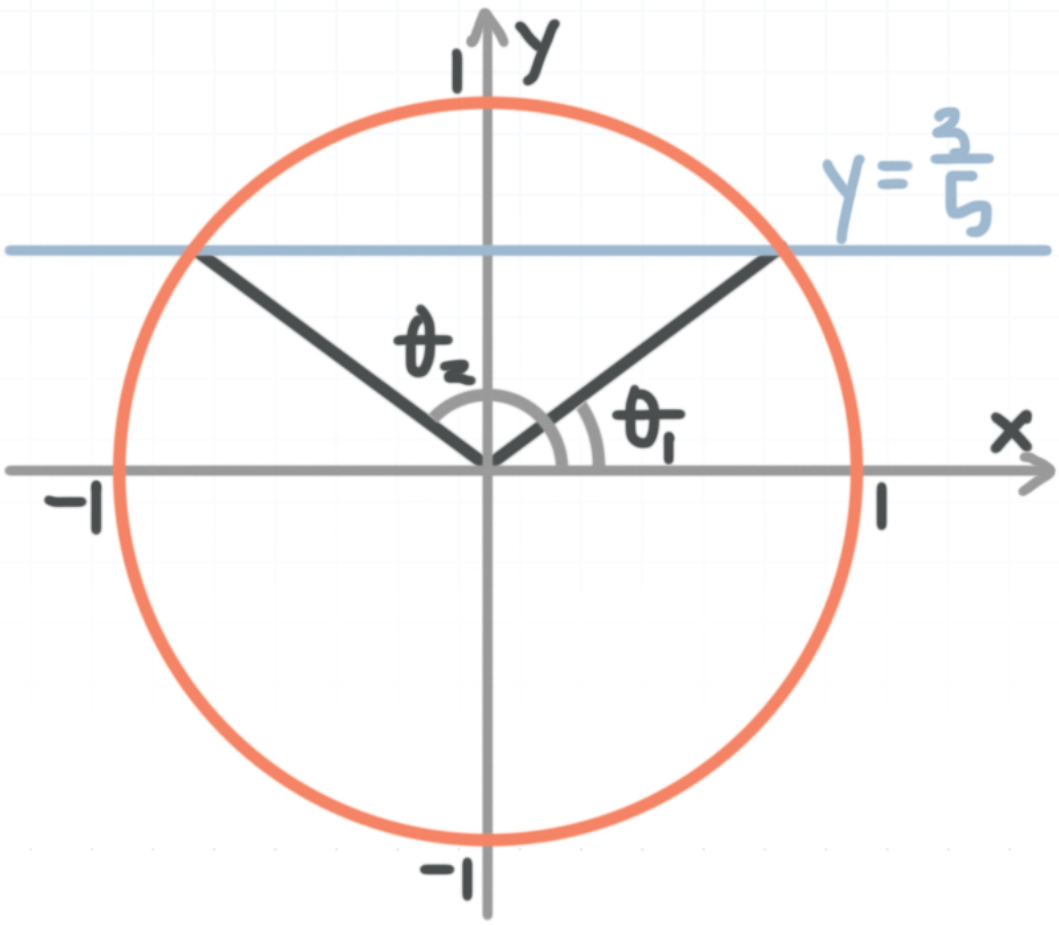
$$\sin(x-4)=\frac{3}{5}$$

Consider the equation $\sin\theta = 3/5$. By the definition, $\sin\theta$ is equal to the y-coordinate of the point P where the angle θ intersects the unit circle. Sketch the unit circle and the line $y = 3/5$. Their intersection points are the solutions of the equation.



The solution in quadrant I is $\theta_1 = \sin^{-1}(3/5)$, and the solution in quadrant II is $\theta_2 = \pi - \sin^{-1}(3/5)$. Extend the solutions to all periods by adding multiples of 2π.

$\theta_1 = \sin^{-1}\frac{3}{5} + 2\pi n$, where n is any integer

$\theta_2 = \pi - \sin^{-1}\frac{3}{5} + 2\pi m$, where m is any integer

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Plug in $\theta = x - 4$ and solve the equations for x.

$$x_1 - 4 = \sin^{-1}\frac{3}{5} + 2\pi n, \text{ where } n \text{ is any integer}$$

$$x_1 = 4 + \sin^{-1}\frac{3}{5} + 2\pi n, \text{ where } n \text{ is any integer}$$

and

$$x_2 - 4 = \pi - \sin^{-1}\frac{3}{5} + 2\pi m, \text{ where } m \text{ is any integer}$$

$$x_2 = 4 + \pi - \sin^{-1}\frac{3}{5} + 2\pi m, \text{ where } m \text{ is any integer}$$

3. Find the domain and range of $f(x)$.

$$f(x) = 4\sin^{-1}(2x - 1) + 3\pi$$

Solution:

The domain of $\sin^{-1} a$ is $[-1,1]$ (because a is equal to the sine of some angle, and the sine function returns values only inside the interval $[-1,1]$). So

$$-1 \leq 2x - 1 \leq 1$$

$$-1 + 1 \leq 2x - 1 + 1 \leq 1 + 1$$

$$0 \leq 2x \leq 2$$

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$$\frac{0}{2} \leq \frac{2x}{2} \leq \frac{2}{2}$$

$$0 \leq x \leq 1$$

So the domain of the function is $[0,1]$.

To find the range of $f(x)$, use the fact that the range of the inverse sine function is $[-\pi/2,\pi/2]$. So

$$\text{Range} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

To get the range of $4\sin^{-1}(2x-1)$, stretch the interval $[-\pi/2,\pi/2]$ by a factor of 4. Then

$$\text{Range} = \left[-4 \cdot \frac{\pi}{2}, 4 \cdot \frac{\pi}{2}\right]$$

$$\text{Range} = [-2\pi,2\pi]$$

To get the range of $4\sin^{-1}(2x-1)+3\pi$, we need to shift the interval up by 3π. So

$$\text{Range} = [-2\pi + 3\pi, 2\pi + 3\pi]$$

$$\text{Range} = [\pi,5\pi]$$

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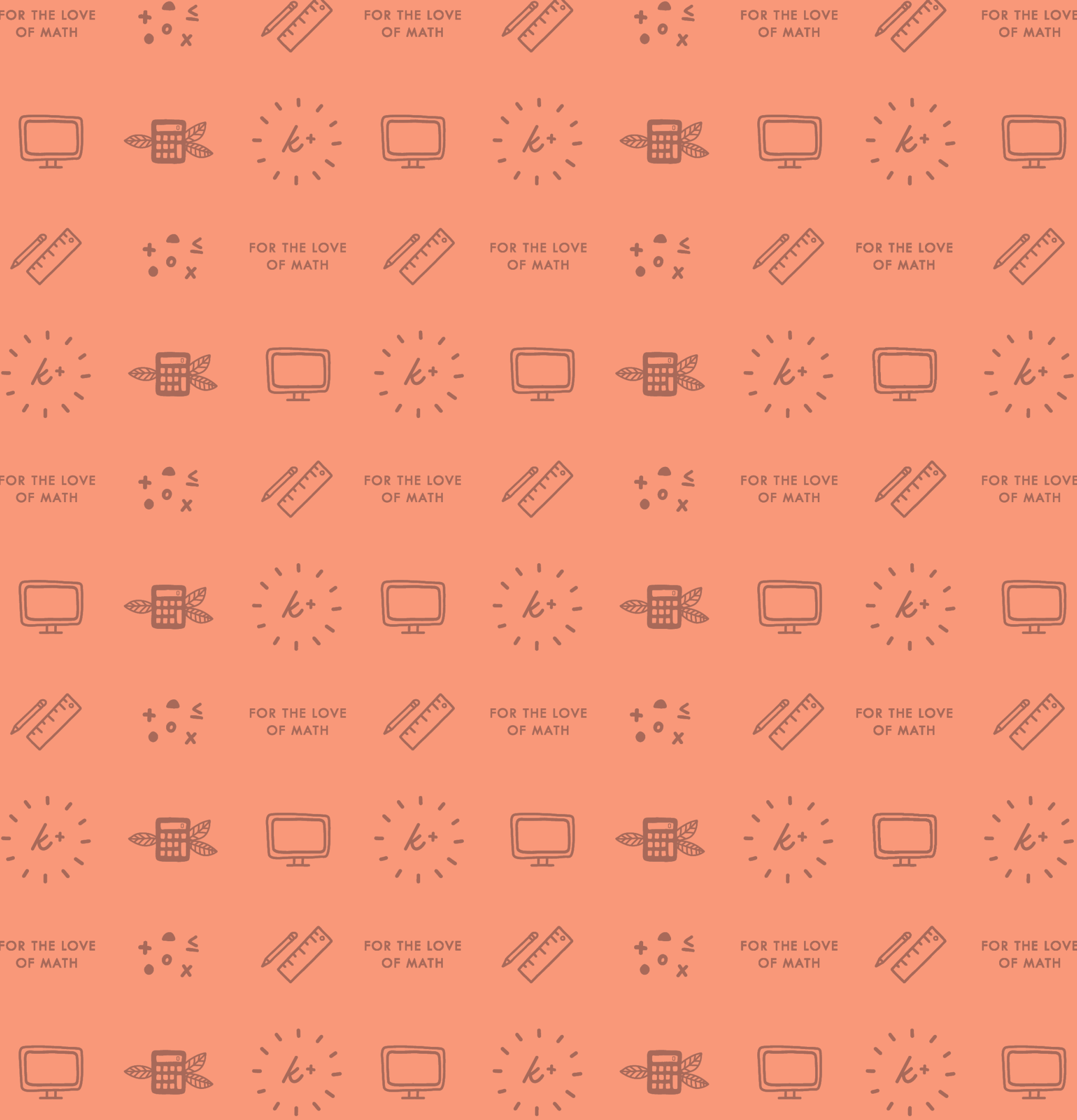
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